Department of Mathematics University of Houston

MATH 4331

Introduction to Real Analysis

Fall 2015

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 7, due Thursday, October 29, 2:30pm

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Does the value $\|(x, y)\| = (|x|^{1/2} + |y|^{1/2})^2$ for $(x, y) \in \mathbb{R}^2$ define a norm on $\mathbb{R}^2$?

Problem 2

Let K be a compact subset of $\mathbb{R}^n$ and let $C(K, \mathbb{R}^m)$ denote the vector space of all continuous functions from K to $\mathbb{R}^m$. Show that if we define $\|f\|_\infty = \sup_{x\in K} \|f(x)\|_2$ for each $f \in C(K, \mathbb{R}^m)$, where $\| \cdot \|_2$ is the Euclidean norm on $\mathbb{R}^m$, then $\| \cdot \|_\infty$ is a norm on $C(K, \mathbb{R}^m)$.

Problem 3

Let $(V, \|\cdot\|)$ be a normed vector space. Assuming convergent sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in V with limits x and y and a sequence $\{\alpha_n\}_{n=1}^\infty$ in $\mathbb{R}$ with limit α, show that $\lim_{n\to\infty}(x_n + y_n) = x + y$ and $\lim_{n\to\infty} \alpha_n x_n = \alpha x$.

Problem 4

Prove that a compact subset K of a normed vector space $(V, \| \cdot \|)$ is complete, meaning each Cauchy sequence in K converges to an element of K.

Problem 5

Prove that a normed vector space $(V, \|\cdot\|)$ is complete if and only if every decreasing sequence of closed balls with radii going to zero has a non-empty intersection. Note that the balls need not be concentric. Hint: Consider the sequence of center points of the balls.