Final Practice Exam – Math 4331/6312
December, 2015

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1 True-False Problems

Put a T in the box next to each statement that is true, an F for each statement that is false.

- [ ] $\{(x,y) \in \mathbb{R}^2 : xy = 1\}$ is a closed set.
- [ ] If $C \subseteq \mathbb{R}^n$ has the property that every sequence has a subsequence that converges to a point in C, then C is closed.
- [ ] If $S = [0,1) \cup (1,2]$, then S is connected.
- [ ] An open subset of a compact set in $\mathbb{R}^n$ has a compact closure.
- [ ] All normed real vector spaces are complete.
- [ ] In a metric space (X,d), the open ball $B_r(x)$, with $x \in X$ and $r > 0$, is never closed.

For the remaining true-false problems, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$

- [ ] If A is compact then $f(A)$ is compact.
- [ ] If a f function is monotonic on $[a,b]$, it is Riemann integrable on $[a,b]$.

In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Let V and W be real vector spaces with two norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively.

Show that $Z = \{(x, y) : x \in V, y \in W\}$, equipped with $\|(x, y)\| = \max\{\|x\|_V, \|y\|_W\}$ is a normed vector space.

3 Problem

Show that if $\mathbb{R}$ and $\mathbb{R}^2$ are equipped with the usual (Euclidean) norms, and K_1 and K_2 are two compact subsets of $\mathbb{R}$, then so is

$$K = \{(x, y) \in \mathbb{R}^2 : x \in K_1, y \in K_2\}.$$

4 Problem

Let A be a connected subset of $\mathbb{R}^n$. Prove that the closure of A is also connected.

5 Problem

Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous, then so is $h = g \circ f, h(x) = g(f(x))$.

6 Problem

Let ℓ^p, $1 \leq p < \infty$ be the normed vector space containing each sequence $a = (a_1, a_2, \dots)$ for which $\|a\|_p = (\sum_{j=1}^{\infty} |a_j|^p)^{1/p} < \infty$. Consider the sequence $\{x_k\}_{k=1}^{\infty}$ with $x_k = (x_{k,1}, x_{k,2}, \dots)$ and

$$x_{k,n} = \begin{cases} 1/n, & n \leq 2^k \\ 0, & n > 2^k \end{cases}$$

show that this sequence does not converge with respect to $\| \cdot \|_p$.

7 Problem

Prove that $\{s_n\}_{n=1}^{\infty}$, $s_n(x) = \sin(nx)$ is not an equicontinuous subset of $C([0, \pi])$.

8 Problem

Show that if f is a real-valued function on the interval $[a, b]$ such that

$$u = \inf\{\int_a^b g(x)dx : g \in C([a, b]), g(x) \geq f(x) \text{ for all } x \in [a, b]\}$$

and

$$l = \sup\{\int_a^b h(x)dx : h \in C([a, b]), h(x) \leq f(x) \text{ for all } x \in [a, b]\}$$

satisfy $u = l$, then f is Riemann integrable.

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