Department of Mathematics University of Houston

MATH 4331/6312

Introduction to Real Analysis

Fall 2017

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 3, due Thursday, September 21, 10am

Please staple this cover page to your homework. Circle your course number, Math 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class **or in the section on compactness and extreme values** in support of your reasoning.

Problem 1

Show that the Cantor set has empty interior. Hint: The construction of the Cantor set can be described in the following way: $C_0 = [0,1]$, $C_1 = [0,1] \setminus (1/3, 2/3)$, $C_2 = C_1 \setminus ((1/9, 2/9) \cup (4/9, 5/9) \cup (7/9, 8/9))$ etc., so at each step open intervals are removed. Note that $(4/9, 5/9)$ appears only for notational convenience.

Problem 2

Let f be defined on $\mathbb{R}$ by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Show that f is continuous at 0 but not continuous at any $a \neq 0$.

Problem 3

Let $m : \mathbb{R}^2 \to \mathbb{R}$ be defined by $m(x, y) = \max\{x, y\}$.

a. Show that m is continuous.

b. Hence, show that for two real-valued continuous functions f and g defined on a set $S \subset \mathbb{R}^n$, the function $h : S \mapsto \mathbb{R}$ defined by $h(x) = \max\{f(x), g(x)\}$ is continuous.

Problem 4

Let A be a compact subset of $\mathbb{R}^n$. Show that for any $x \in \mathbb{R}^n$, there is $a \in A$ which is closest to x among the points in A, so for any $y \in A$, $\|y - x\| \geq \|a - x\|$. Hint: Fix x and introduce a useful function on A which you show to be continuous, then quote a result from class.

Problem 5

Assume a real-valued function f is continuous on $\mathbb{R}^n$ and satisfies $f(x) \geq 0$ for all $x \in \mathbb{R}^n$ as well as $\lim_{\|x\| \to \infty} f(x) = 0$, i.e. for each $\epsilon > 0$ there is $R > 0$ such that $f(x) < \epsilon$ for all x with $\|x\| > R$. Show that f attains its maximum value.