Department of Mathematics University of Houston

MATH 4331/6312

Introduction to Real Analysis
Fall 2017

First name: ______________________ **Last name:** ________________________ **Points:**

Assignment 7, due Thursday, November 2, 10am

Please staple this cover page to your homework. Circle your course number, 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class or on the handout on the Fundamental Theorem of Calculus in support of your reasoning.

Problem 1

Let $a < b < c$ and f be a bounded function on $[a, c]$ that is Riemann integrable on $[a, b]$ and on $[b, c]$. Show that f is Riemann integrable on $[a, c]$ and that

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \,.$$

Problem 2

Does the value $\|(x, y)\| = (|x|^{1/2} + |y|^{1/2})^2$ for $(x, y) \in \mathbb{R}^2$ define a norm on $\mathbb{R}^2$? Explain your answer by supporting it with facts.

Problem 3

Let $(V, \|\cdot\|))$ be a normed vector space. Assuming convergent sequences $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ in V with limits x and y and a sequence $\{\alpha_n\}_{n=1}^\infty$ in $\mathbb{R}$ with limit α, show that $\lim_{n\to\infty}(x_n + y_n) = x + y$ and $\lim_{n\to\infty}\alpha_n x_n = \alpha x$.

Problem 4

Let K be a compact subset of $\mathbb{R}^n$ and let $C(K, \mathbb{R}^m)$ denote the vector space of all continuous functions from K to $\mathbb{R}^m$. Show that if we define $\|f\|_\infty = \sup_{x\in K} \|f(x)\|_2$ for each $f \in C(K, \mathbb{R}^m)$, where $\|f(x)\|_2$ is the Euclidean norm of $f(x) \in \mathbb{R}^m$, then $\|f\|_\infty$ defines a norm on $C(K, \mathbb{R}^m)$.

Problem 5

Let c_0 be the vector space of all convergent sequences $x = \{x_n\}_{n=1}^\infty$ in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = 0$. Let $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

1. Show that for $x \in c_0$ there is $k \in \mathbb{N}$ such that $|x_k| = \|x\|_\infty$.
2. Show that $x \mapsto \|x\|_\infty$ defines a norm on c_0.