Department of Mathematics University of Houston

MATH 4331/6312

Introduction to Real Analysis
Fall 2017

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Tuesday, November 21, 10am

Please staple this cover page to your homework. Circle your course number, 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Prove that a compact subset K of a normed vector space $(V, \|\cdot\|)$ is complete, meaning each Cauchy sequence in K converges to an element of K.

Problem 2

Let c_0 be the vector space of all convergent sequences $x = \{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = 0$. Let $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$. Consider a Cauchy sequence $\{x_k\}_{k=1}^{\infty}$ in c_0 whose elements are denoted by $x_k = \{x_{k,n}\}_{n=1}^{\infty}$.

a. Show that for each fixed $n \in \mathbb{N}$, $\{x_{k,n}\}_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$.

b. Deduce that the limit $y_n = \lim_{k\to\infty} x_{k,n}$ defines $y = \{y_n\}_{n=1}^{\infty}$ with $\|y\|_\infty < \infty$.

c. Show that $\lim_{k\to\infty} \|x_k - y\|_\infty = 0$

d. Conclude that y belongs to c_0 and thus c_0 is complete.

Problem 3

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of all rational numbers in $\mathbb{Q} \cap [0,1]$. For $f, g \in C([0,1])$, let

$$\langle f, g\rangle = \sum_{n=1}^{\infty} 2^{-n} f(r_n) g(r_n)\,.$$

Show that this defines a (positive definite) inner product on the space $C([0,1])$.

Problem 4

If $f \in C([0,1])$ and $1 \le r \le s < \infty$, show that $\|f\|_1 \le \|f\|_r \le \|f\|_s \le \|f\|_\infty$. Hint: Use Hölder's inequality with $g(x) = 1$ and exponent $p = s/r$. Hence, show that if $\{f_n\}_{n=1}^{\infty}$ in $C([0,1])$ converges uniformly to $f \in C([0,1])$, then the sequence also converges with respect to the norm $\|\cdot\|_p$ for any $1 \le p < \infty$.