# MATH 4331/6312

## Introduction to Real Analysis

Fall 2017

**First name:** ____________________ **Last name:** ______________________ **Points:**

# Assignment 9, due Thursday, November 30, 10am

**Please staple this cover page to your homework. Circle your course number, 4331 or 6312.** When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

## Problem 1

A normed vector space $V$ is strictly convex if $\|u\| = \|v\| = \|(u+v)/2\| = 1$ for vectors $u, v \in V$ implies that $u = v$.

1. Show that an inner product space with the norm induced by the inner product, meaning $\|x\| = (\langle x, x\rangle)^{1/2}$ for each $x \in V$, is always strictly convex.
2. Show that the plane $\mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$ is not strictly convex.

## Problem 2

Prove that a normed vector space $(V, \|\cdot\|)$ is complete if and only if every nested decreasing sequence of closed balls $\overline{B}_{r_1}(a_1) \supset \overline{B}_{r_2}(a_2) \supset \cdots$ with radii $r_j$ going to zero as $j \to \infty$ has a non-empty intersection $\cap_{j=1}^{\infty} \overline{B}_{r_j}(a_j)$. Note that the balls need not be concentric. Hint: Consider the sequence of center points of the balls.

## Problem 3

Find all intervals on which the sequence of real-valued functions $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}$ defined by $f_n(x) = \frac{x^{2n}}{n+x^{2n}}$ converges uniformly. Explain the reasons supporting your answer.

## Problem 4

Show that if the sequence of numbers $\{a_n\}_{n=1}^{\infty}$ satisfies $\sum_{n=1}^{\infty} |a_n| < \infty$, then the series $\sum_{n=1}^{\infty} a_n \cos(nx)$ converges uniformly on $[0, 2\pi]$. This means, the partial sums

$$s_N(x) = \sum_{n=1}^{N} a_n \cos(nx)$$

define a sequence of functions $\{s_N\}_{N=1}^{\infty}$ that converges uniformly on $[0, 2\pi]$. Hint: First show that the sequence is Cauchy with respect to $\|\cdot\|_{\infty}$.