The Fundamental Theorem of Calculus

Proposition 1. *Let $a \leq \hat{x} \leq b$, $f : [a, b] \to \mathbb{R}$ be a bounded function that is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, \hat{x}]$ and if g is the function*

$$g(x) = \begin{cases} f(x), & x \in [a, \hat{x}] \\ 0, & \textit{else} \end{cases}$$

then

$$\int_a^{\hat{x}} f(x)dx = \int_a^b g(x)dx\,.$$

Proof. The statement on integrability of f on $[a, \hat{x}]$ is equivalent to g being integrable on $[a, \hat{x}]$. Given $\epsilon > 0$, then by the integrability of f on $[a, b]$, there is a partition $\mathcal{P}$ of $[a, b]$ with $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$. Let $\mathcal{Q} = \mathcal{P} \cup \{\hat{x}\}$, then $\mathcal{Q}$ is a refinement of $\mathcal{P}$ and so

$$U(f, \mathcal{Q}) - L(f, \mathcal{Q}) \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Moreover, if $Q = \{x_0, x_1, \ldots, x_m = \hat{x}, \ldots, x_n\}$ then from g vanishing outside of $[x_0, x_m]$, we have

$$U(g, \mathcal{Q}) - L(g, \mathcal{Q}) \leq U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < \epsilon.$$

This means g is integrable on $[a, b]$. Again noting that subintervals outside of $[x_0, x_m] = [a, \hat{x}]$ do not contribute, we let $\mathcal{S} = \mathcal{Q} \cap [x_0, \hat{x}]$, then

$$U(g, \mathcal{S}) - L(g, \mathcal{S}) \leq U(g, \mathcal{Q}) - L(g, \mathcal{Q}) < \epsilon\,.$$

Thus, the integrability condition holds for g on $[a, \hat{x}]$, or equivalently, for f on $[a, \hat{x}]$.

For the identity between the integrals, let $\mathcal{S}$ be a partition of $[a, \hat{x}]$ and $\mathcal{P}$ a partition of $[a, b]$, $\mathcal{Q} = \mathcal{S} \cup \mathcal{P}$, and $\mathcal{R} = \mathcal{Q} \cap [a, \hat{x}]$, then from g vanishing outside of $[a, \hat{x}]$,

$$L(f, \mathcal{R}) = L(g, \mathcal{R}) = L(g, \mathcal{Q}) \geq L(g, \mathcal{P})$$

and

$$U(f, \mathcal{R}) = U(g, \mathcal{R}) = U(g, \mathcal{Q}) \leq U(g, \mathcal{P})\,.$$

Moreover, $\mathcal{R}$ is a refinement of $\mathcal{S}$, so taking the supremum of $L(f, \mathcal{S})$ over all $\mathcal{S}$ is the same as taking the supremum of $L(f, (\mathcal{S} \cup \mathcal{P}) \cap [a, \hat{x}]\}$. By comparing with $L(g, \mathcal{P})$, this gives $\sup_{\mathcal{S}} L(f, \mathcal{S}) \geq \sup_{\mathcal{P}} L(g, \mathcal{P})$. Similarly, $\inf_{\mathcal{S}} U(f, \mathcal{S}) \leq \inf_{\mathcal{P}} U(g, \mathcal{P})$. By integrability, $\sup_{\mathcal{P}} L(g, \mathcal{P}) = \inf_{\mathcal{P}} U(g, \mathcal{P}) = \int_a^b g(x)dx$, so $\sup_{\mathcal{S}} L(f, \mathcal{S}) = \inf_{\mathcal{S}} U(f, \mathcal{S}) = \int_a^{\hat{x}} f(x)dx = \int_a^b g(x)dx$.

□

Theorem 2 (Fundamental Theorem of Calculus). *Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$, then for $x \in [a, b]$,*

$$F(x) = \int_a^x f(t)dt$$

defines a continuous function. Moreover, if f is continuous at x_0 in $[a, b]$, then $F'(x_0) = f(x_0)$.

Proof. We first note that if $\sup_{a \leq x \leq b} |f(x)| \leq M$, then for $a \leq x < y \leq b$,

$$\int_a^y f(t)dt - \int_a^x f(t)dt = \int_a^b \chi_{[a,y]}(t) f(t)dt - \int_a^b \chi_{[a,x]}(t) f(t)dt\,.$$

Using the linearity of integrals then allows us to combine the difference in one integral

$$\int_a^y f(t)dt - \int_a^x f(t)dt = \int_a^b (\chi_{[a,y]}(t) - \chi_{[a,x]}(t)) f(t)dt\,.$$

Choosing a partition $\mathcal{P} = \{a, x, y, b\}$ then gives with $g(t) = \chi_{[a,y]}(t) - \chi_{[a,x]}(t)) f(t)$,

$$-M(y - x) \leq L(g, \mathcal{P}) \leq \int_a^y f(t)dt - \int_a^x f(t)dt \leq U(g, \mathcal{P}) \leq M(y - x)\,.$$

This shows that F is Lipschitz continuous.

For the differentiability, let $a < x_0 < x_0 + h < b$, then using the continuity of f shows that for any $\epsilon > 0$, there is $\delta > 0$ such that if $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \epsilon$.

Let $h > 0$, then

$$\begin{aligned}\frac{1}{h}(F(x_0+h) - F(x_0)) - f(x_0) &= \frac{1}{h}\int_a^{x_0+h}(1-\chi_{[a,x_0]})(t)f(t)dt - f(x_0)\\ &= \frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)dt\\ &= \frac{1}{h}\int_{x_0}^{x_0+h}(f(t)-f(x_0))dt\,.\end{aligned}$$

Next, if $0 < h < \delta$, then from $|f(t) - f(x_0)| < \epsilon$, we get

$$|\frac{1}{h}(F(x_0+h) - F(x_0)) - f(x_0)| \leq \frac{1}{h}\int_{x_0}^{x_0+h}|f(t)-f(x_0)|dt < \epsilon\,.$$

This implies $\lim_{h\to 0^+}(\frac{1}{h}(F(x_0+h) - F(x_0)) - f(x_0)) = 0$. The case $h < 0$ is handled similarly by writing

$$\frac{1}{h}(F(x_0+h) - F(x_0)) = -\frac{1}{-h}(F(x_0) - F(x_0+h))$$

and expressing the right-hand side as an integral on the interval $[x_0 + h, x_0]$. □

Corollary 3. *Let $f : [a, b] \to \mathbb{R}$ be continuous. If $F : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable on (a, b) and $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof. Let $G(x) = \int_a^x f(t)dt + F(a)$, then for $a < x < b$, we have that $G'(x) = F'(x)$ and $G(a) = F(a)$. Hence, $G(x) = F(x)$ for $a < x < b$. Since $|G(b) - G(x)| = |\int_x^b f d\alpha| \leq K(\alpha(b) - \alpha(x))$, where $K = \sup\{|f(x)| : a \leq x \leq b\}$, we see that $G(b) = \lim_{x\to b^-} G(x) = \lim_{x\to b^-} F(x) = F(b)$, since F is continuous. Therefore, $F(b) = G(b) = \int_a^b f(x)dx + F(a)$ and the formula follows. □