Department of Mathematics University of Houston

MATH 4331/6312

Introduction to Real Analysis
Fall 2019

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 1, due Thursday, August 29, 10am

Please staple this cover page to your homework. Circle your course number, 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 0

Prove that if a sequence $\{p_1, p_2, \ldots\}$ in $\mathbb{R}$ converges to $\lim_{n\to\infty} p_n = p$, then it is bounded, that is, there exists a $L \geq 0$ such that for all $n \in \mathbb{N}$, $|p_n| \leq L$. Hint: Start with: Convergence of $\{p_n\}_{n\in\mathbb{N}}$ means that for every $\epsilon > 0$, ... Now *choose any* $\epsilon > 0$, then split the set $\mathbb{N}$ into two subsets and show the boundedness of the sequence when the index is taken from either one of those two subsets.

Problem 1

Show that the sequence $\{x_1, x_2, \ldots\}$ defined by $x_1 = 1$ and $x_{k+1} = \frac{1}{2}\left(x_k + \frac{3}{x_k}\right)$ converges to $\sqrt{3}$. Hint: Assume we already know $1 \leq \sqrt{3} \leq 2$. Re-express the relationship between x_k and x_{k+1} as

$$\frac{x_{k+1} - \sqrt{3}}{x_{k+1} + \sqrt{3}} = \left(\frac{x_k - \sqrt{3}}{x_k + \sqrt{3}}\right)^2.$$

Problem 2

Suppose x and y are unit vectors in $\mathbb{R}^n$. Show that if $\|\frac{1}{2}(x+y)\| = 1$, then $x = y$.

Problem 3

Let $x_0 = (a_0, b_0)$ in $\mathbb{R}^2$, with $0 < a_0 < b_0$, and define inductively for each $n \in \mathbb{N}$, $(a_{n+1}, b_{n+1}) = (\sqrt{a_n b_n}, (a_n + b_n)/2)$.

a. Show, using induction, that for each $n \in \mathbb{N}$, $0 < a_n < a_{n+1} < b_{n+1} < b_n$.

b. Estimate $b_{n+1} - a_{n+1}$ in terms of $b_n - a_n$.

c. Show that there is $c \in \mathbb{R}$ such that $\lim_{n\to\infty}(a_n, b_n) = (c, c)$.

Problem 4

Show that if a set A in $\mathbb{R}^n$ is closed, then it is complete, meaning any Cauchy sequence $\{p_1, p_2, \ldots\}$ in A converges and has a limit in A.