MATH 4331/6312

Introduction to Real Analysis
Fall 2019

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 10, due Thursday, November 21, 8:30am

Please staple this cover page to your homework. Circle your course number, 4331 or 6312. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let V and W be real vector spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Consider the vector space $Z = \{(v, w) : v \in V, w \in W\}$. Show that $\|(x, y)\| \equiv \max\{\|x\|_V, \|y\|_W\}$ defines a norm on Z.

Problem 2

Prove that a normed vector space $(V, \|\cdot\|)$ is complete if and only if every nested decreasing sequence of closed balls $\overline{B}_{r_1}(a_1) \supset \overline{B}_{r_2}(a_2) \supset \cdots$ with radii r_j going to zero as $j \to \infty$ has a non-empty intersection $\cap_{j=1}^{\infty} \overline{B}_{r_j}(a_j)$. Note that the balls need not be concentric. Hint: Consider the sequence of center points of the balls.

Problem 3

Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of all rational numbers in $\mathbb{Q} \cap [0, 1]$. For $f, g \in C([0, 1])$, let

$$\langle f, g \rangle = \sum_{n=1}^{\infty} 2^{-n} f(r_n) g(r_n) \,.$$

Show that this defines a (positive definite) inner product on the space $C([0, 1])$.

Problem 4

A normed vector space V is strictly convex if $\|u\| = \|v\| = \|(u+v)/2\| = 1$ for vectors $u, v \in V$ implies that $u = v$.

1. Show that an inner product space with the norm induced by the inner product, meaning $\|x\| = (\langle x, x \rangle)^{1/2}$ for each $x \in V$, is always strictly convex.
2. Show that the plane $\mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$ is not strictly convex.