**MATH 4332**

**Introduction to Real Analysis**

**Spring 2016**

**First name:** ____________________ **Last name:** ______________________ | **Points:** |

# Assignment 6, due Thursday, March 24, 2:30pm

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

## Problem 1

Let $f(x) = x^3 - 2$. Explain why $x^* = 2^{1/3}$ is the unique (real) root of $f$. Show that $1.25 < 2^{1/3} < 1.26$. Use Newton's method to compute $2^{1/3}$ to eight decimal places (8 correct digits following the decimal point), starting from $x_1 = 1.25$. Using a calculator or a software package for computations is encouraged, however only basic arithmetic is allowed.

## Problem 2

Let $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ be $C^\infty$, that is, all iterated partial derivatives with respect to the two variables are continuous on $[a, b] \times \mathbb{R}$, and $T$ be defined as before with an initial value $y_0$.

Show that for any $f_1 \in C([a, b])$, $f_n \equiv T^{n-1} f_1$ has $n - 1$ continuous derivatives. Use this to conclude that the unique solution $f$ to the differential equation $f'(x) = h(x, f(x))$ with initial value $f(a) = y_0$ is arbitrarily often continuously differentiable on $[a, b]$.

## Problem 3

Let $h_1$ and $h_2$ be real-valued Lipschitz-continuous functions on $[a, b] \times \mathbb{R}$ and $f_1$ and $f_2$ be solutions of $f_1'(x) = h_1(x, f_1(x))$ and $f_2'(x) = h_2(x, f_2(x))$ with initial values $y_0 = f_1(a) = f_2(a)$. Show that if $h_1(x, y) \le h_2(x, y)$ for each $(x, y) \in [a, b] \times \mathbb{R}$, then $f_1(x) \le f_2(x)$ for each $x \in [a, b]$.

## Problem 4

Find the Taylor polynomials of order 3 at the point $a$ for the choices of the function $f$ given below and estimate the error of approximating $f$ by its Taylor polynomial at the point $b$.

1. $f(x) = \tan x$, $a = \pi/4$, $b = 3/4$;
2. $f(x) = \sqrt{1 + x^2}$, $a = 0$, $b = 0.1$;
3. $f(x) = x^4$, $a = 1$, $b = 0.99$.

## Problem 5

Use the convergence of the Taylor series of $\sin(x)$ about $x = \pi/6$ to approximate $\sin(31°)$ to 10 decimal places.