MATH 4332

Introduction to Real Analysis

Spring 2016

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 9, due Tuesday, May 3, 2:30pm

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Recall that from the results in class, $|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\cos((2j-1)x)$ for each $x \in [-\pi,\pi]$ where the series converges uniformly and also with respect to the metric induced by the norm $\|\cdot\|_2$, with $\|f\|_2 = \left(\int_{-\pi}^{\pi}|f(x)|^2dx\right)^{1/2}$.

a. Prove that if f is bounded and Riemann integrable on $[-\pi,\pi]$ and $S_N f$ denotes the N-th partial sum of the Fourier series, then $\lim_{N\to\infty}\|S_N f\|_2^2 = \|f\|_2^2$.

b. Use this result together with the concrete choice of $f(x) = |x|$ to show that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}\,.$$

Hint: Split the sum into two sums, over even and odd k.

Problem 2

Let g and h be real-valued functions on $\mathbb{R}$ such that g is differentiable at a and h is differentiable at b. Show that $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x_1,x_2) = g(x_1)h(x_2)$ is differentiable at $x_0 = (a,b)$.

Problem 3

Let $f:\mathbb{R}^n\to\mathbb{R}$ be given by $f(x) = \frac{\|x\|^4}{1+\|x\|^2}$ Use the chain rule to show that f is differentiable at each $x\in\mathbb{R}^n$ and compute $Df(x)$.

Problem 4

Let $f:\mathbb{R}^2\to\mathbb{R}^2, f(x,y) = (x^2-y^2, 2xy)$. For which $(x,y)\in\mathbb{R}^2$ is there a ball $B_\epsilon(x,y)$ with some $\epsilon > 0$ so that f restricted to this ball has an inverse? Explain your answer.