MATH 4332/6313

Introduction to Real Analysis

Spring 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 4, due Thursday, February 22, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $X = [0, \infty)$, equipped with the usual metric from $\mathbb{R}$. Show that the function $f(x) = \sqrt{x}$ is uniformly continuous on X. Hint: Use the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for any $a, b \geq 0$ (proved by squaring both sides).

Problem 2

Show that if (X, d) and (Y, ρ) are two metric spaces, X is compact and $f : X \to Y$ is continuous, then f is uniformly continuous.

Problem 3

Let (X, d) and (Y, ρ) be two metric spaces with completions (C, d) and (D, ρ), where we assume $X \subset C$ and $Y \subset D$. Define a metric σ on $C \times D$ by $\sigma((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2)$. Show that the completion of the metric space $(X \times Y, \sigma)$ is $(C \times D, \sigma)$.

Problem 4

Show that $[0, 1]$, equipped with the usual metric, is not the disjoint union of a countably infinite family of nonempty closed sets $\{A_n\}_{n=1}^{\infty}$. Hint: If $U_n = A_n^\circ$, then explain why the complement of $\cup_{n=1}^{\infty} U_n$, $X = [0, 1] \setminus \cup_{n=1}^{\infty} U_n$ is complete (with respect to the usual metric). Next, find an integer n_0 and an open interval V such that $X \cap V \neq \emptyset$ and $X \cap V \subset A_{n_0}$. Show $U_n \cap V = \emptyset$ if $n \neq n_0$.