Department of Mathematics University of Houston

MATH 4332/6313
Introduction to Real Analysis
Spring 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 6, due Thursday, March 22, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $f(x) = x^2 - 5$. Show that f has a root x^* somewhere in the interval $[2, 3]$. Calculate Newton's $g(x)$ and prove that g maps $[2, 3]$ into $[2, 3]$, with $g'(x) \le \frac{1}{2}$ for $x \in [2, 3]$. Prove that if we perform Newton's method with $x_1 = 2$, then $|x_n - x^*| \le \frac{1}{2^n}$.

Problem 2

Let $f(x) = x - \cos(x)$ so if x^* is a root for f, then $\cos(x^*) = x^*$. Compute Newton's $g(x)$ and find concrete numbers a and b with $0 \le a \le b \le 1$ such that g maps $[a, b]$ into $[a, b]$ and it is a contraction mapping. How does the Lipschitz constant of g compare with the one we had in class when we discussed the fixed point for $\cos x$?

Problem 3

Let $f(x) = x^3 - 2$. Explain why $x^* = 2^{1/3}$ is the unique (real) root of f. Show that $1.25 < 2^{1/3} < 1.26$. Use Newton's method to compute $2^{1/3}$ to eight decimal places (8 correct digits following the decimal point), starting from $x_1 = 1.25$. Using a calculator or a software package for computations is encouraged, however only basic arithmetic is allowed.

Problem 4

Let $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ be C^∞, that is, all iterated partial derivatives with respect to the two variables are continuous on $[a, b] \times \mathbb{R}$, and T be defined as before with an initial value y_0.

Show that for any $f_1 \in C([a, b])$, $f_n \equiv T^{n-1} f_1$ has $n - 1$ continuous derivatives. Use this to conclude that the unique solution f to the differential equation $f'(x) = h(x, f(x))$ with initial value $f(a) = y_0$ is arbitrarily often continuously differentiable on $[a, b]$.

Problem 5

Let h_1 and h_2 be real-valued Lipschitz-continuous functions on $[a, b] \times \mathbb{R}$ and f_1 and f_2 be solutions of $f_1'(x) = h_1(x, f_1(x))$ and $f_2'(x) = h_2(x, f_2(x))$ with initial values $y_0 = f_1(a) = f_2(a)$. Show that if $h_1(x, y) \le h_2(x, y)$ for each $(x, y) \in [a, b] \times \mathbb{R}$, then $f_1(x) \le f_2(x)$ for each $x \in [a, b]$.