MATH 4332/6313
Introduction to Real Analysis
Spring 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Thursday, April 19, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

1. If $x_0, x_1, \ldots x_n$ are distinct points in $[A, B]$ and $a_0, a_1, \ldots a_n$ are in $\mathbb{R}$, show that there is a unique polynomial p_a of degree at most n such that $p_a(x_j) = a_j$ for each j. Hint: Find polynomials q_j such that $q_j(x_k) = 0$ for each $k \neq j$ and $q_j(x_j) = 1$. You may express them in factorized form.

2. Next, show that there is a constant M such that for all $a = (a_0, a_1, \ldots, a_n)$, $\|p_a\|_\infty \leq M\|a\|_2$.

Problem 2

Suppose that $f \in C([a, b])$, $\epsilon > 0$ and $x_1, x_2, \ldots x_n$ are points in $[a, b]$. Prove that there is a polynomial p such that $p(x_i) = f(x_i)$ and $\|f - p\|_\infty < \epsilon$. Hint: First approximate f closely by some polynomial of sufficiently high degree, then use the result of the previous problem to adjust this approximation.

Problem 3

Let Q_n be the space of polynomials of maximal degree n such that each $p \in Q_n$ satisfies $p(0) = p(1) = 0$. Let $f \in C([0, 1])$, $f(0) = f(1) = 0$. Explain why among all $p \in Q_n$, there is a minimizer for the function $E : p \mapsto \|p - f\|_\infty$.

Problem 4

If $f \in C([-1, 1])$ is an even (odd) function, then show that the best approximation among the polynomials of degree n is also even (odd).

Problem 5

Assume $f \in C([a, b])$ is twice continuously differentiable and $f''(x) > 0$ on $[a, b]$. Show that the best linear approximation (a polynomial of degree one) p to f has the slope $p'(x) = (f(b) - f(a))/(b - a)$.