MATH 4332/6313
Introduction to Real Analysis
Spring 2018

First name: __________________ **Last name:** ____________________ **Points:**

Assignment 9, due Thursday, April 26, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Compute the Fourier series of the following functions on $[-\pi, \pi]$:

a. $f(t) = \cos^3(t)$,

b. $f(t) = |\sin t|$,

c. $f(t) = t$.

Problem 2

Show that the Fourier series for $f(x) = x^2$ on $[-\pi, \pi]$ is given by

$$\frac{\pi^2}{3} - 4\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos(kx)}{k^2}$$

and assuming (uniform) convergence, together with an appropriate choice of x, show that

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

Problem 3

Recall that from the results in class, $|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{(2j-1)^2}\cos((2j-1)x)$ for each $x \in [-\pi, \pi]$ where the series converges uniformly and also with respect to the metric induced by the norm $\|\cdot\|_2$, with $\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx\right)^{1/2}$.

a. Prove that if $f \in C_{\text{per}}([-\pi, \pi])$ and $S_N f$ denotes the N-th partial sum of the Fourier series, then $\lim_{N\to\infty}\|S_N f\|_2^2 = \|f\|_2^2$.

b. Use this result together with the concrete choice of $f(x) = |x|$ to show that

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}.$$

Hint: Split the sum into two sums, over even and odd k. Simplify.