Practice Exam 2 – Math 4332/6313
March, 2018

First name: ________________ Last name: ________________ Last 4 digits student ID: ________

In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

1 Problem

Let $f(x) = \frac{1}{2}e^{-x} - \frac{1}{4}$.

1. Briefly explain why the function f has a single root in the interval $[0, 1]$.

2. Set up Newton's method to find the root x^* of f. Show that Newton's map is a contraction on $[0, 1]$.

3. Starting with $x_1 = 0$, compute successive approximations x_2 and x_3 for x^*. Give an estimate for $|x_3 - x^*|$.

2 Problem

Consider the ODE $f'(x) = 1 + \frac{1}{2}f(x)$ with initial value $f(0) = 1$ on $[0, 1]$.

1. Use an integral to define the map $T : C([0, 1]) \to C([0, 1])$ as in class.

2. Show that T is a contraction on $C([0,1])$, equipped with the (usual) max-metric.

3. Starting with $f_1(x) = 1$, compute $f_3 = T^2 f_1$. If f^* is the unique solution to the ODE, estimate the distance $\|f_3 - f^*\|_\infty$.

3 Problem

Show that if (X, d) is a compact metric space and $f : X \to X$ satisfies $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ then f has a fixed point.

4 Problem

Find the Taylor polynomial of order 2 of the function $f(x) = \sin(x)$, $a = \pi/2$. Estimate the error when $f(3\pi/5)$ is replaced by $P_2(3\pi/5)$.

5 Problem

Show that if f is an even function in $C([-1,1])$, then for any $\epsilon > 0$ there is an even polynomial p, that is, $p(x) = p(-x)$ for $x \in [-1,1]$, such that $\|f - p\|_\infty < \epsilon$. Hint: If q is any polynomial, then $p(x) = \frac{1}{2}(q(x) + q(-x))$ is an even polynomial.

[empty page]