Practice Final Exam – Math 4332
April, 2018

First name: ____________________ Last name: ____________________ Last 4 SID: _____________

1 True-False Problems

Put T beside each statement that is true, F beside each statement that is false.

☐ If (X, d) and (Y, ρ) denote arbitrary metric spaces, X is complete and $f : X \to Y$ is continuous and onto, then Y is complete.

☐ If a function $f : [-\pi, \pi] \to \mathbb{R}$ is bounded and Riemann integrable, then its Fourier series is uniformly convergent.

For the remaining true-false problems, all spaces are Euclidean spaces, $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$

☐ If A is compact then $f(A)$ is compact.

☐ If A is closed then $f(A)$ is closed

☐ If A is bounded then $f(A)$ is bounded.

☐ If f has Lipschitz constant 1, then it has a unique fixed point.

In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Let (X, d) and (Y, ρ) be metric spaces. Show that $Z = \{(x, y) : x \in X, y \in Y\}$, equipped with $\gamma : Z \to \mathbb{R}$ by $\gamma((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2)$ is a metric space.

3 Problem

Suppose X, Y are metric spaces and $f, g : X \to Y$ are continuous functions. Prove that the set $A = \{x \in X : f(x) = g(x)\}$ is closed in X.

4 Problem

Let $\mathbb{R}$ and $\mathbb{R}^2$ be equipped with the usual (Euclidean) metrics. Prove that if K_1 and K_2 are two compact subsets of $\mathbb{R}$, then the set

$$K = \{(x, y) : x \in K_1, y \in K_2\}$$

is compact in $\mathbb{R}^2$.

5 Problem

Let (X, d) be a metric space. Prove that a sequence $\{p_n\}_{n\in\mathbb{N}}$ is convergent if and only if it is Cauchy and it has a convergent subsequence.

6 Problem

Let $f : [-1, 1] \to \mathbb{R}$ be continuously differentiable and assume there is $0 \leq \epsilon < 1/2$ such that $|f'(x) - 1| \leq \epsilon$ for all $x \in [-1, 1]$.

(a) Show that for any x, u such that $x, x + u \in [-1, 1]$, $|f(x + u) - f(x) - u| \leq |u|/2$.

(b) With the additional assumption $f(0) = 0$, show that $|x - f(x)| \leq |x|/2$.

(c) Again with the additional assumption $f(0) = 0$, show that for each fixed $y \in [-1/2, 1/2]$, the function $g_y(x) = y + x - f(x)$ is a contraction mapping on $[-1, 1]$.

(d) With the assumption as in (b) and (c), show that there exists $h : \mathbb{R} \to \mathbb{R}$ such that $h(f(x)) = x$ for all $x \in [-1/4, 1/4]$.

7 Problem

Let $f(x) = e^x$. Find the Taylor series of f at $a = 1$ and state its radius of convergence.

8 Problem

Show that if f is continuous on $[0, 1]$ and

$$\int_0^1 f(x)dx = 0\,,$$

then there is a sequence of polynomials $(p_n)_{n=1}^{\infty}$ such that $p_n \to f$ uniformly on $[0, 1]$ and for each $n \in \mathbb{N}$, $\int_0^1 p_n(x)dx = 0$.

9 Problem

Let $f(x) = x$ on $[-\pi, \pi]$.

1. Compute the Fourier coefficients of the function f.

2. Use the identity between the squared norm of the function and a series involving the Fourier coefficients to compute $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

10 Problem

Find the closest linear function p to $f(x) = x^3$ on $[-1, 1]$, meaning $p(x) = ax + b$ for $a, b \in \mathbb{R}$ and $\|f - p\|_\infty$ is minimal among all such linear functions in $C([-1, 1])$. Hint: Recall the properties of Chebyshev polynomials and the facts $T_0(x) = 1$, $T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 2$.

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