Department of Mathematics University of Houston

MATH 4332/6313

Introduction to Real Analysis

Spring 2020

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 1, due Thursday, January 23, 8:30am

Please staple this cover page to your homework. Circle your course number, 4332 or 6313. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Find all intervals on which the sequence of real-valued functions $(f_n)_{n=1}^{\infty}$ on $\mathbb{R}$ defined by $f_n(x) = \frac{x^{2n}}{n+x^{2n}}$ converges uniformly. Explain the reasons supporting your answer.

Problem 2

Show that if the sequence of numbers $(a_n)_{n=1}^{\infty}$ satisfies $\sum_{n=1}^{\infty} |a_n| < \infty$, then the series $\sum_{n=1}^{\infty} a_n \cos(nx)$ converges uniformly on $[0, 2\pi]$. This means, the partial sums

$$s_N(x) = \sum_{n=1}^{N} a_n \cos(nx)$$

define a sequence of functions $\{s_N\}_{N=1}^{\infty}$ that converges uniformly on $[0, 2\pi]$. Hint: First show that the sequence is Cauchy with respect to $\|\cdot\|_\infty$.

Problem 3

If $f \in C([0,1])$ and $1 \le r \le s < \infty$, show that $\|f\|_1 \le \|f\|_r \le \|f\|_s \le \|f\|_\infty$. Hint: Use Hölder's inequality with $g(x) = 1$ and exponent $p = s/r$. Hence, show that if $(f_n)_{n=1}^{\infty}$ in $C([0,1])$ converges uniformly to $f \in C([0,1])$, then the sequence also converges with respect to the norm $\|\cdot\|_p$ for any $1 \le p < \infty$.

Problem 4

Prove that the family $\mathcal{F} = \{f_n : f_n(x) = \sin(nx), n \in \mathbb{N}, x \in [0,\pi]\}$ is not an equicontinuous family in $C([0,\pi])$.