Department of Mathematics University of Houston

MATH 4332/6313

Introduction to Real Analysis
Spring 2020

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 2, due Thursday, January 30, 8:30am

Please staple this cover page to your homework. Circle your course number, 4332 or 6313. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let c_0 be the space containing each sequence $x = (x_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = 0$, equipped with the norm $\|x\|_\infty = \sup_n |x_n|$. Show that the closed unit ball $\overline{B}_1(0)$ in c_0 is not compact.

Problem 2

Let

$$\mathcal{F} = \left\{ g \in C([0,1]) : \text{ there is } f \in C([0,1]), \|f\|_\infty \le 1 \text{ with } g(x) = \int_0^x f(t)dt \text{ for any } x \in [0,1] \right\}.$$

We wish to find the closure of this set.

a. Show that the closure of $\mathcal{F}$ in $C([0,1])$ contains all functions with Lipschitz constant at most 1 and the property $f(0) = 0$. Hint: For any such Lipschitz-continuous function f, construct a sequence with elements $f_n \in \mathcal{F}$ such that

$$f_n(2^{-n}k) = \left(1 - \frac{1}{n+1}\right) f(2^{-n}k)$$

holds for any $0 \le k \le 2^n$. It may be useful to first interpolate $\left(1 - \frac{1}{n+1}\right) f$ with a piecewise linear, continuous function that is not in $\mathcal{F}$ and then modify it to make it continuously differentiable.

b. Show that if f has Lipschitz constant > 1, or is not Lipschitz continuous, then it is not in the closure of $\mathcal{F}$. Hint: Use an indirect proof and the Mean Value Theorem.

Problem 3

Let K be a compact subset of $\mathbb{R}^n$ and $\mathcal{F}$ an equicontinuous family of functions in $C(K)$, where K is compact. If for each $x \in K$, $\sup_{f\in\mathcal{F}} |f(x)| = M_x < \infty$, prove that $\sup\{\|f\|_\infty : f \in \mathcal{F}\} < \infty$. Hint: Use equicontinuity to establish the bound $\sup\{|f(x)| : f \in \mathcal{F}, x \in B_\delta(a)\} \le M_a + 1$ for some fixed $a \in K$. Given $|f_n(x_n)| \to \infty$ for some sequences of $f_n \in \mathcal{F}$ and $x_n \in K$, extract a convergent subsequence of $(x_n)_{n=1}^{\infty}$ to derive a contradiction.