MATH 4332/6313

Introduction to Real Analysis

Spring 2020

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 6, due Thursday, March 26, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that the completion of a metric space (X, ρ) is compact if and only if X is totally bounded.

Problem 2

Let (X, d) and (Y, ρ) be two metric spaces with completions (C, d) and (D, ρ), where we assume $X \subset C$ and $Y \subset D$. Define a metric σ on $C \times D$ by $\sigma((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \rho(y_1, y_2)$. Show that the completion of the metric space $(X \times Y, \sigma)$ is $(C \times D, \sigma)$.

Problem 3

Let (X, d), (Y, ρ) and (Z, σ) be metric spaces and $f : X \to Y$ be a Lipschitz continuous map with (Lipschitz) constant $L \geq 0$, $g : Y \to Z$ a Lipschitz continuous map with constant $S \geq 0$. Prove that the composition $h = g \circ f : X \to Z$ has Lipschitz constant LS.

Problem 4

Let (X, ρ) be a metric space. Recall that a *uniformly* continuous function $f : X \to \mathbb{R}$ extends uniquely to a (uniformly) continuous $g : C \to \mathbb{R}$, where C is the completion of X and $X \subset C$, so $g(x) = f(x)$ for each $x \in X$. Explain why (nevertheless) *any* continuous function g on $\mathbb{R}$ is uniquely characterized by its restriction $f : \mathbb{Q} \to \mathbb{R}, f(x) = g(x)$.