MATH 4332/6313

Introduction to Real Analysis

Spring 2020

First name: ____________________ **Last name:** ______________________ | **Points:** |

Assignment 7, due Thursday, April 2, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that $f(x) = \sin(x)$ is not a contraction mapping on $[-1, 1]$.

Problem 2

Let $f(x) = \frac{x}{2} + \frac{1}{x}$. Use some basic calculus to show that f maps $[1, 2]$ into $[1, 2]$, and use the mean value theorem to show that it is a contraction mapping. What is the value of the unique fixed point x^*? If you choose $x_0 = \frac{3}{2}$ as your starting value, estimate $|x^* - x_n|$ for $n \in \mathbb{N}$ with the help of the distance bound in the contraction mapping theorem.

Problem 3

Let $f(x) = x^2 - 5$. Show that f has a root x^* somewhere in the interval $[2, 3]$. Calculate Newton's $g(x)$ and prove that g maps $[2, 3]$ into $[2, 3]$, with $g'(x) \leq \frac{1}{2}$ for $x \in [2, 3]$. Prove that if we perform Newton's method with $x_0 = 2$, then $|x_n - x^*| \leq \frac{1}{2^n}$.

Problem 4

Let $f(x) = x - \cos(x)$ so if x^* is a root for f, then $\cos(x^*) = x^*$. Compute Newton's $g(x)$ and find concrete numbers a and b with $0 \leq a \leq b \leq 1$ such that g maps $[a, b]$ into $[a, b]$ and it is a contraction mapping. How does the Lipschitz constant of g compare with the one we had in class when we discussed the fixed point for $\cos x$?

Problem 5

Let $f(x) = x^3 - 2$. Explain why $x^* = 2^{1/3}$ is the unique (real) root of f. Show that $1.25 < 2^{1/3} < 1.26$. Use Newton's (improved) method to compute $2^{1/3}$ to eight decimal places (8 correct digits following the decimal point), starting from $x_0 = 1.25$. Using a calculator or a software package for computations is encouraged, however only basic arithmetic is allowed.