MATH 4332/6313

Introduction to Real Analysis

Spring 2020

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Thursday, April 9, 8:30am

Please staple this cover page to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $b > 0$ and $a \in \mathbb{R}$ be fixed and define for $f \in C([0, b])$ and $0 \le x \le b$,

$$\Phi(f)(x) = a + \int_0^x f(t) x e^{-xt} dt\,.$$

Prove that Φ is a contraction mapping on $C([0, b])$. Use this result to conclude that the integral equation

$$f(x) = a + \int_0^x f(t) x e^{-xt} dt$$

has a unique solution on the (unbounded) interval $[0, \infty)$.

Problem 2

Show that the initial value problem $y' = xy$, $y(0) = 1$ has a unique solution f on $[0, 1/2]$. Starting with $f_0(x) = 1$, compute a distance estimate between f_n, given by $f_n = \Phi(f_{n-1})$, and f in $C([0, 1/2])$.

Problem 3

Let $h : [a, b] \times \mathbb{R} \to \mathbb{R}$ be C^∞, that is, all iterated partial derivatives with respect to the two variables are continuous on $[a, b] \times \mathbb{R}$, and Φ be defined as before with an initial value y_0.

Show that for any $f_0 \in C([a, b])$, $f_n \equiv \Phi^n f_0$ has n continuous derivatives. Use this to conclude that the unique solution f to the differential equation $f'(x) = h(x, f(x))$ with initial value $f(a) = y_0$ is arbitrarily often continuously differentiable on $[a, b]$.

Problem 4

Consider the initial value problem $y' = 1 + y^2$ and $y(0) = 0$. Can you guess the solution or use methods you learnt in a prerequisite course to solve it?

a. Show that the map Φ as defined before is not a contraction mapping on $C([0, b])$ for *any* $b > 0$.

b. Find $b > 0$ so that Φ maps the closed unit ball in $C([0, b])$ to itself and is a contraction mapping on this ball.

c. Conclude the existence and uniqueness of solutions to the initial value problem.