UNIVERSITY OF HOUSTON
MIDTERM EXAMINATION

Term: Spring   Year: 2009

Student: First Name _____ Last Name _____

UH Student ID Number _____

Course Abbreviation and Number	Math 4355
Course Title	Mathematics of Signal Representations
Section(s)	001
Sections of Combined Course(s)	
Section Numbers of Combined Course(s)	
Instructor(s)	B. G. Bodmann

Date of Exam	March 11, 2009
Time Period	Start time: 5:30pm End time: 7:30pm
Duration of Exam	2 hours
Number of Exam Pages (including this cover sheet)	11 pages
Exam Type	Closed book
Additional Materials Allowed	None

Marking Scheme:

Question	Points	Score
1	30	
2	20	
3	20	
4	30	
5	20	
Total	120	

1. (30 points) Let V_0 be the sub-space of $L^2([-\frac{1}{2}, \frac{1}{2}])$ spanned by the set of functions $\{g_1, g_2\}$, where

$$g_1(x) = 1$$

and

$$g_2(x) = x \, .$$

   (a) (10 points) Is $\{g_1, g_2\}$ an orthonormal basis for V_0? Compute all the relevant quantities to explain your answer.

(b) (10 points) Let

$$f(x) = \begin{cases} 3, & -1/2 \le x < 0 \\ 1, & 0 \le x \le 1/2 \,. \end{cases}$$

Express the orthogonal projection $\hat{f}$ of f onto V_0 in terms of a linear combination of g_1 and g_2.

(c) (5 points) Compute the value $\hat{f}(x)$ for $x \in [-1/2, 1/2]$.

(d) (5 points) Sketch the graphs of f and of $\hat{f}$ from the preceding part on the interval $[-1/2, 1/2]$. You can put them in the same coordinate system.

2. (20 points) Let V be the vector space of real-valued polynomials of degree at most two,

$$V = \{p(x) = c_0 + c_1 x + c_2 x^2, c_0, c_1, c_2 \in \mathbb{R}\},$$

and define for two functions $p, q \in V$ the "dot" product

$$\langle\langle p, q \rangle\rangle = p(0)q(0) + p'(0)q'(0) + \frac{1}{4}p''(0)q''(0).$$

(a) (8 points) Determine all $p \in V$ for which $\langle\langle p, p \rangle\rangle = 0$.

(b) (12 points) Is this dot product an inner product? If this is true, explain briefly why. If not, explain which of the properties of inner products is violated.

3. (20 points) Let a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $[0, 1]$ be defined by

$$f_n(x) = \begin{cases} \sin(nx), & 0 \le x \le \pi/n \\ 0, & \text{else} . \end{cases}$$

(a) (10 points) Show that $f_n \to 0$ in $L^2([0, 1])$ as $n \to \infty$.

(b) (10 points) Does $f_n \to 0$ uniformly on $[0, 1]$ as $n \to \infty$? Why/why not?

4. (30 points) Consider the function $f(x) = \cos(x/2)$ on $x \in [-\pi, \pi]$.

    (a) (15 points) Find the Fourier coefficients a_k and b_k (real form of the Fourier series) for f. Simplify as much as possible, eliminating any trigonometric functions in your answer, if you have time.

(b) (5 points) Sketch three periods of the function to which the Fourier series converges.

(c) (10 points) Is the Fourier series uniformly convergent to f on the interval $[-\pi, \pi)$? Quote an appropriate theorem and use it to justify your answer.

5. (20 points) Let $g(x) = x\sin(x)$ on $-\pi \le x < \pi$.

   (a) (10 points) For which points $x \in [-\pi, \pi)$ does the Fourier series converge to $g(x)$? Justify your answer by referring to conditions for pointwise convergence.

   (b) (10 points) Assuming that you have (correctly) computed the partial sum of the Fourier series for g as

   $$S_N(x) = 1 - \frac{1}{2}\cos(x) - \sum_{k=2}^{N} \frac{2}{k^2-1}\cos(k\pi)\cos(kx),$$

   find the value for the series

   $$\sum_{k=2}^{\infty} \frac{1}{k^2-1}$$

   by choosing an appropriate x for which the Fourier series converges.

Trig formulas

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\sin^2(\alpha) = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$$

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

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