First Name: ______________________

Last Name: ______________________

Signature: ______________________

Student I.D. No.: ______________________

Math 6320 Practice Final Exam

December, 2010
Two hours and twenty minutes

University of Houston

Instructions:

1. Put your name, signature and I.D. No. in the blanks above.
2. There are **four questions** in this exam. Answer the questions in the spaces provided, using the backs of pages or the blank pages at the end for overflow or rough work.
3. Your grade will be influenced by how clearly you present your solutions. **Justify your solutions carefully** by referring to definitions and results from class where appropriate.
4. **This is a closed book exam.**

1. Let (X, M, μ) be a measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions.

   (a) Define almost everywhere convergence of the sequence and define convergence in $L^1(\mu)$.

   (b) Prove that any sequence which is Cauchy with respect to the metric induced by the norm on $L^1(\mu)$ has an almost everywhere converging subsequence.

2. Recall that a Borel measure μ on the Borel algebra B of a topological space X is regular on a set $A \in B$ if for all $\epsilon > 0$ there exists a compact set K and an open set V with $K \subset A \subset V$ such that $K \subset A \subset V$ and $\mu(V \setminus K) < \epsilon$. If X is a compact metric space then show that any finite Borel measure μ on (X, B) is regular, meaning it is regular on all $A \in B$.

   Suggestion: Start by showing that the class of sets on which μ is regular is a σ-algebra.

3. Let (X, M, μ) be a measure space. Let S be the class of complex-valued measurable simple functions which are non-zero on a set of finite measure. If $1 \leq p < \infty$, show that S is dense in $L^p(\mu)$.

4. Consider a Hilbert space H and a subspace M.

   (a) Prove that $M^\perp$, the orthogonal complement of M, is a closed subspace.

   (b) Prove that the closure of M is identical to $(M^\perp)^\perp$.