Math 6320 - Practice Midterm Exam - Fall 2010

1. Let (X, M, μ) be a measure space, let $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ be two sequences of **non-negative** measurable functions and assume $f_n(x) \leq g_n(x)$ for all $n \in \mathbb{N}$, as well as point-wise convergence $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$, for all $x \in X$. Assume that all $\int g_n d\mu$ and $\int g d\mu$ are finite, and that $\int g_n d\mu \to \int g d\mu$.

   (a) Quote a famous result from class to establish that $\liminf_n \int f_n d\mu \geq \int f d\mu$.

   (b) Prove $\limsup_n \int f_n d\mu \leq \int f(x) d\mu$. Hint: consider $h_n = g_n - f_n$.

   (c) What can you say about $\lim_n \int f_n d\mu$?

2. Let (X, M, μ) be a measure space and let $0 < c < \infty$.

   (a) Let $f \geq 0$ be a measurable function on X such that $\int_E f d\mu \leq c\mu(E)$ for all $E \in M$. If $\mu(X) < \infty$, prove that $\mu(\{x \in X : f(x) > c\}) = 0$.

   (b) Let $g \geq 0$ be a measureable function on X such that $\mu(\{x \in X : g(x) > c\}) = 0$. Prove that then $\int_E g d\mu \leq c\mu(E)$ for all $E \in M$.

3. Let μ be a regular Borel measure on a **compact** Hausforff space X and assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ such that $\mu(K) = 1$ and $\mu(H) < 1$ for every compact, proper subset H of K (this means for each compact $H \subset K$ with $H^c \cap K \neq \emptyset$).

4. State Lusin's theorem (without proving it).