First Name: ______________________

Last Name: ______________________

Signature: ______________________

Student I.D. No.: ______________________

Math 6321 Practice Final Exam

May, 2011
Three hours

University of Houston

Instructions:

1. Put your name, signature and I.D. No. in the blanks above.
2. There are **five questions** in this exam. Answer the questions in the spaces provided, using the backs of pages or the blank pages at the end for overflow or rough work.
3. Your grade will be influenced by how clearly you present your solutions. **Justify your solutions carefully** by referring to definitions and results from class where appropriate.
4. **This is a closed book exam.**

1. (a) State the definition of absolute continuity for a complex-valued function f on an interval $I = [a, b] \subset \mathbb{R}$, $a < b$.

(b) State a fundamental theorem which relates the values of $f(x)$ and $f(a)$ of an absolutely continuous function f on $[a, b]$ to an expression involving an integral over $[a, x]$ for any $a \leq x \leq b$.

2. If we have measure spaces (X, M, μ) and (Y, N, ν), and an M-measurable μ-integrable function f on X, as well as an N-measurable ν-integrable g on Y, and we let F be the function on $X \times Y$ with values $F(x, y) = f(x)g(y)$, show that F is measurable with respect to the product algebra $M \times N$ and integrable with respect to the product measure $\mu \times \nu$. You may quote results from class to support your answer.
   Hint: First prove that the function $G : X \times Y \to \mathbb{C}^2$ given by $G(x, y) = (f(x), g(y))$ is measurable. You may use that the Borel algebra on $\mathbb{C}^2$ is generated by open rectangles. Then show that F is measurable by referring to a result from class, and refer to another result to explain the integrability.

3. Let $1 < p < \infty$ and let $q = p/(p-1)$. Assume that $\sum_{n \in \mathbb{N}} a_n b_n$ is absolutely convergent for each $b \in \ell^q(\mathbb{N})$. Prove that then $a \in \ell^p(\mathbb{N})$.

4. Let (X, M, μ) be a σ-finite measure space, and Λ a bounded linear functional on $L^1(\mu)$.

   (a) Show that there is an essentially bounded function g such that

   $$\Lambda(f) = \int fg d\mu$$

   for each $f \in L^1(\mu)$.

   (b) Show that such a g satisfies $\|\Lambda\| = \|g\|_\infty$.

5. Assume that f is a continuous function which is Lebesgue integrable on $\mathbb{R}$ and define for $x \in \mathbb{R}, t \in [0, 2\pi]$ and $n \in \mathbb{N}$,

$$F_n(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-n}^{n} f(x + m/(2\pi))e^{-imt}\,.$$

(a) Show that the functions $\{F_n\}_{n\in\mathbb{N}}$, when restricted to the rectangle $\mathbb{D} = [0, \frac{1}{2\pi}] \times [0, 2\pi]$ in $\mathbb{R}^2$, form a sequence converging in $L^1(\mathbb{D})$.

(b) Show that $f(x)$ is for any $x \in \mathbb{R}$ obtained from F by the inverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{[0,2\pi]} F(x,t)dt$$

(c) Show that if f is in addition square integrable on $\mathbb{R}$ with L^2-norm $\|f\|$, then

$$\|f\|^2 = \int_{\mathbb{D}} |F(x,t)|^2 dxdt\,.$$