MATH 6360

Applied Analysis

Fall 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 1, due Friday, August 31, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that the function $T : \mathbb{R} \to \mathbb{R}$ given by

$$T(x) = \frac{\pi}{2} + x - \arctan(x)$$

does not have a fixed point. Prove that for all $x, y \in \mathbb{R}$,

$$|T(x) - T(y)| < |x - y| .$$

Why does this example not contradict the contraction mapping theorem?

Problem 2

Let $f(x) = \frac{x}{2} + \frac{1}{x}$. Use some basic calculus to show that f maps $[1, 2]$ into $[1, 2]$, and use the mean value theorem to show that it is a contraction mapping. What is the value of the unique fixed point x^*? If you choose $x_0 = \frac{3}{2}$ as your starting value, estimate $|x^* - x_n|$ for $n \in \mathbb{N}$.

Problem 3

For $b > 0$ and $a \in \mathbb{R}$, define T on $C[0, b]$ by $Tf(x) = a + \int_0^x f(t)xe^{-xt}dt$. Prove that T is a contraction. Hence show that there is a unique solution $f \in C([0, \infty))$ to the integral equation $f(x) = a + \int_0^x f(t)xe^{-xt}dt$.

Problem 4

Consider the initial value problem with the differential equation $y'(x) = 1 + xy(x)$ and $y(0) = 0$.

1. Show that for any $0 < b < 1$, the integral operator T associated with this differential equation is a contraction mapping on $C([-b, b])$.

2. Show that there is a unique solution of this differential equation on $[-b, b]$ for this initial value and *any* $b < \infty$. Hence deduce that there is a unique solution of the initial value problem on $\mathbb{R}$.

Problem 5

An $n \times n$ real matrix A is said to be diagonally dominant if for each row the sum of the absolute value of off-diagonal terms in this row is strictly less than the value of the diagonal term in this row. Write $A = D - L - U$, where D is a diagonal matrix, L lower triangular and U upper triangular (both L and U with zero diagonal). Let the row-sum norm $\|X\|_\infty$ of a matrix $X = (X_{i,j})_{i,j=1}^n$ be defined by

$$\|X\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |X_{i,j}| \,,$$

then show that if A is diagonally dominant, $\|L+U\|_\infty < \|D\|_\infty$. If A is diagonally dominant, prove that A is invertible and the solution to $Ax = b$ can be obtained as the limit of the sequence $(x_n)_{n=1}^\infty$ obtained from

$$x_{n+1} = D^{-1}(L+U)x_n + D^{-1}b \,.$$

Hint: Use the metric on $\mathbb{R}^n$ given by $d(x,y) = \max_{1 \le i \le n} |x_i - y_i|$.