MATH 6360

Applied Analysis

Fall 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 2, due Friday, September 7, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $p \in \mathbb{N}$, $b > 0$ and assume u is the solution of the integral equation

$$u(x) = \int_0^x \sin(u(t))(u(t))^p dt$$

on the interval $[-b, b]$.

1. Let $M = \sup_{-b \le x \le b} |u(x)|$. Prove that for each integer $n \ge 0$, $|u(x)| \le M^{np}|x|^n/n!$. Hint: $|\sin(y)| \le |y|$.
2. Use the preceding part to show that $u = 0$.

Problem 2

Consider the initial value problem

$$y'(x) = x^2 + (y(x))^2, y(0) = 0$$

1. Show that this differential equation satisfies a local Lipschitz condition (in the second variable) on the set $Q = [0, b] \times [-R, R]$, but not on the set $[0, b] \times \mathbb{R}$.
2. Integrate the inequality $y'(x) \ge 1 + (y(x))^2$ to prove that the solution to the initial value problem grows above any bound in finite time.

Problem 3

Let y be the solution to the initial value problem $y'(x) = e^{xy(x)}$ and $y(0) = 1$ for $x \in [-1/2, 1/2]$. Suppose you wish to compare this with the solution y_n to the initial value problem $y'(x) = \sum_{k=0}^{n} \frac{(xy(x))^k}{k!}$, $y_n(0) = 1$, on $[-1/2, 1/2]$.

1. Show that as $n \to \infty$, $y_n \to y$ uniformly on $[-1/2, 1/2]$.
2. Find n so that $d(y, y_n) < 0.0001$.

Problem 4

Let y be a solution of the initial value problem $y'(x) = h(x, y(x))$ and $y(a) = y_0$, where h is continuous on $[a, b] \times \mathbb{R}$ and L-Lipschitz in the second variable. Assume η is a differentiable function satisfying $|\eta'(x) - h(x, \eta(x))| \leq \epsilon$ for each $x \in [a, b]$ and $|\eta(a) - y_0| \leq \delta$. Show that for $x \in [a, b]$,

$$|y(x) - \eta(x)| \leq \delta e^{L(x-a)} + \frac{\epsilon}{L}(e^{L(x-a)} - 1)\,.$$

Hint: Find a variation of the proof for stability of solutions.