MATH 6360

Applied Analysis

Fall 2018

First name: ____________________ **Last name:** ________________________ **Points:**

Assignment 3, due Friday, September 14, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $a > 0$ and consider the integral equation for $f : [-a, a] \to \mathbb{R}$,

$$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x-y)^2} f(y) dy\,.$$

Show that it has a unique non-negative solution in $C([-a, a])$.

Problem 2

Let $h : [a, b] \times \mathbb{R}$ be a continuous function and for each fixed $x \in [a, b]$, $y \mapsto h(x, y)$ is non-increasing in y.

1. Let f and g be two solutions to the differential equation $y'(x) = h(x, y(x))$ with any initial values. Show that $\tau(x) = |f(x) - g(x)|$ is non-increasing in x. Hint: If $f(x) > g(x)$ on some interval I and $(x_1, x_2) \subset I$, express $f(x_2) - g(x_2) - (f(x_1) - g(x_1))$ as an integral.
2. Use the preceding part to show that if the initial value problem with $f(a) = y_0$, $y_0 \in \mathbb{R}$, has a solution on $[a, b]$, then it is unique.

Problem 3

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))$ with $f_1(x, y, z) = 3x + 2y + z$, $f_2(x, y, z) = 2xy + z^2$, $f_3(x, y, z) = 4xyz$. Compute the derivative $Df(x_0, y_0, z_0)$ at $(x_0, y_0, z_0) = (1, 2, 0)$. Use the inverse function theorem to linearize the local inverse of f at $(7, 4, 0)$ and find an approximate solution to $f(x, y, z) = (7.1, 4.2, 0.3)$.

Problem 4

Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x, y) = (e^x \cos y, e^x \sin y)\,.$$

Show for that for $(x_0, y_0) \in \mathbb{R}^2$, f is locally invertible in an open ball centered at (x_0, y_0). Give an explicit example for an open set U containing (x_0, y_0) on which the restriction of f is invertible.