Department of Mathematics University of Houston

MATH 6360
Applied Analysis
Fall 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 4, due Friday, September 21, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Use the implicit function theorem to show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 11 \\ x - y + 2z + u &= -2 \\ 2x + 2y - 3z + 2u &= 15 \end{aligned}$$

can be solved locally for (y, z, u) as function of x. Give the formulas for the derivatives dy/dx, dz/dx, and du/dx. The point $(x_0, y_0, z_0, u_0) = (1, 3, -1, 2)$ solves the equations. Use the derivatives to give a linear approximation to the nearby point $(0.9, y, z, u)$ that satisfies the equations.

Problem 2

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuously differentiable. Assume at the point (x_0, y_0, z_0) the partial derivatives satisfy $\partial f(x_0, y_0, z_0)/\partial x \neq 0$, $\partial f(x_0, y_0, z_0)/\partial y \neq 0$ and $\partial f(x_0, y_0, z_0)/\partial z \neq 0$. Explain that there are functions g_i with $x = g_1(y, z)$, $y = g_2(x, z)$ and $z = g_3(x, y)$ that are defined in some open sets in $\mathbb{R}^2$ and $f(g_1(y, z), y, z) = f(x, g_2(x, z), z) = f(x, y, g_3(x, y)) = f(x_0, y_0, z_0)$ and if (x, y, z) has coordinates in these open sets, then

$$\frac{\partial g_1}{\partial y} \frac{\partial g_2}{\partial z} \frac{\partial g_3}{\partial x} = -1.$$

Verify this identity for the partial derivatives for the example of a van der Waals gas with $f(p, V, T) = pV - RT = f(p_0, V_0, T_0)$, where $p_0, V_0, V_0 > 0$ and $R > 0$ is a constant.

Problem 3

Let y be implicitly given as a function of x by $f(x, y) = e^y + y^3 + x^3 + x^2 = 1$. Explain why this equation defines $y = g(x)$ globally on $\mathbb{R}$. Find the values of x for which $g'(x) = 0$. Verify that if $g'(x) = 0$, then

$$g''(x) = -\frac{1}{\partial f(x, y)/\partial y} \frac{\partial^2 f(x, y)}{\partial x^2} .$$

Use this to decide for which x the function g has a local maximum/minimum.

Problem 4

Let A be a symmetric, real $n \times n$ matrix, so $A_{i,j} = A_{j,i}$ for $i, j \in \{1, 2, \ldots, n\}$. Consider the corresponding quadratic function $f : \mathbb{R}^n \to \mathbb{R}, f(x) = \sum_{i,j=1}^{n} A_{i,j} x_i, x_j$. Show that on the sphere $\{x \in \mathbb{R}^n : \|x\| = 1\}$ in $\mathbb{R}^n$, f assumes its maximum at a point x for which $Ax = \lambda x$ with some $\lambda \in \mathbb{R}$.