MATH 6360

Applied Analysis

Fall 2018

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Friday, November 9, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Explain why $C([0,1])$, equipped with the metric d_p coming from the L^p-norm for $1 \leq p \leq \infty$ is separable. Hint: Use the result of a prior homework problem to take care of $p = \infty$.

Problem 2

Let X be a Banach space, Y a normed vector space and $T : X \to Y$ bounded, linear. Assume there is $C > 0$ such that for each $x \in X$, $\|Tx\| \geq C\|x\|$. Show that the range $T(X)$ forms a complete subspace of Y and that the map $T' : X \to T(X), T'(x) = T(x)$ has a bounded inverse.

Problem 3

Let $X = c_{0,0}$, the space of sequences with finitely many non-zero elements, equipped with the norm from ℓ^∞. Let $T : X \to X$ be given by $(Tx)_k = x_k/k, k \in \mathbb{N}$. Show that T is a bijection, but that it does not have a bounded inverse.

Problem 4

Let $X = C([0,1])$ be equipped with d_∞. We define $T : X \to X$ by $(Tf)(x) = \int_0^x f(t)dt$. Show that T is injective. Describe $T(X)$. Does $T' : X \to T(X)$ have a bounded inverse?

Problem 5

Let $F : C([-1,1]) \to \mathbb{R}$ be given by $F(f) = \int_0^1 f(t)dt - \int_{-1}^0 f(t)dt$ where $C([-1,1])$ is equipped with d_∞. Let $Y = \ker F = \{f \in C([-1,1]), F(f) = 0\}$ and $h(x) = x$, then show that $\inf_{y \in Y} \|y - h\| = \frac{1}{2}$ but that there is no $z \in Y$ with $\|z - h\| = \frac{1}{2}$. Hint: How does $|F(f)|/\|F\|$ relate to the distance between f and Y?