Final Practice Exam – Math 6360
December, 2018

First name: ________________ Last name: ____________________ Last 4 digits ID: _____________

1 Memorization

a. State the contraction mapping theorem on metric spaces.

b. State a theorem on the existence and uniqueness of initial-value problems of first-order differential equations on an interval $[a, b]$.

c. State carefully the definition of the open mapping theorem for linear transformations $T : X \to Y$, with appropriate space X and Y.

d. State under which condition the norm on a normed real vector space is induced by an inner product.

Turn in this first part to obtain the next portion of the exam.

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In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Let $X = C([0, a])$, equipped with the metric induced by the norm $\|f\|_\infty = \sup_{0 \le x \le a} |f(x)|$.

a. Show that the averaging operator A defined by $Af(0) = f(0)$ and

$$Af(x) = \frac{1}{x}\int_0^x f(x)\,dx\,,\ x > 0$$

is a bounded linear operator mapping X to itself with $\|A\| \le 1$.

b. Does A have any fixed points? If so, find one. If not, explain why not.

3 Problem

Let $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$. Find the maximum value of $\|(x, y, z)\|_p = (|x|^p + |y|^p + |z|^p)^{1/p}$ for $1 < p < 2$ occuring among all points in B. Distinguish cases if necessary.

4 Problem

Let the set $A \subset \ell^\infty$ be given by

$$A = \{x = (x_1, x_2, x_3, \dots) \in \ell^\infty : 0 < x_n < 1, n \in \mathbb{N}\}.$$

Decide whether this set is open in ℓ^∞, equipped with the metric induced by the sup-norm. Explain the reasons for your answer.

5 Problem

Is the sequence of functions $(f_n)_{n=1}^{\infty}$ in $C([0,1])$, equipped with the sup-norm, given by

$$f_n(t) = \sqrt{n}(t^n - t^{n+1})$$

convergent in $C([0,1])$?

6 Problem

Let X be a uniformly convex normed linear space, $F \in X^*$, $F \not\equiv 0$. Consider the set $Y = \{y \in X : F(y) = 1\}$. Prove that

$$\|F\| = \frac{1}{\inf_{y \in Y} \|y\|} \,.$$

7 Problem

Let H be the Hilbert space ℓ^2 and S the map defined by

$$(Sx)_k = x_k/k, k \in \mathbb{N}$$

for any $x = (x_1, x_2, \dots) \in \ell^2$. Show that S is a bounded map from ℓ^2 to ℓ^2, and that it is one-to-one. Is the range of S a closed subspace of ℓ^2? Explain your answer.

8 Problem (20 points)

Let X be the space $C^1([0,1])$ containing all continuously differentiable functions. Show that the two norms

$$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$$

and

$$\|f\|_0 = |f(0)| + \|f'\|_\infty$$

are equivalent.

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