MATH 6360

Applicable Analysis
Fall 2021

First name: ____________________ **Last name:** ________________________ **Points:**

Assignment 1, due Thursday, September 2, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $\mathbb{Z}_2 = \{0, 1\}$ and for $n \in \mathbb{N}$, the set $X = \mathbb{Z}_2^n$ contain all $x = (x_1, x_2, \ldots, x_n)$ with entries $x_j \in \mathbb{Z}_2$, $1 \le j \le n$. Define a metric d on X by $d(x, y) = \sum_{j=1}^n |x_j - y_j|$.

a. Show that for any $x \in X$, the closed ball $\overline{B_r}(x)$ of radius $r \in \mathbb{N}$, centered at x, has at least $(n/r)^r$ elements. (Hint: First explain why the size of the ball is larger than $\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1}$.)

b. Show that for $r \in \mathbb{N}$, if x and x' have distance $d(x, x') > 2r$, then $\overline{B_r}(x)$ and $\overline{B_r}(x')$ are disjoint.

c. Let $r \in \mathbb{N}$ and $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ have mutual distance $d(x^{(i)}, x^{(i')}) \ge 2r+1$ for all $i \ne i'$. By counting elements, prove that $k \le \left(\frac{r}{n}\right)^r 2^n$.

Problem 2

Explain why the function $T : \mathbb{R} \to \mathbb{R}$ given by

$$T(x) = \frac{\pi}{2} + x - \arctan(x)$$

does not have a fixed point. Prove that for all $x, y \in \mathbb{R}$,

$$|T(x) - T(y)| < |x - y| .$$

Why does this example not contradict the contraction mapping theorem?

Problem 3

For $b > 0$ and $a \in \mathbb{R}$, define T on $C([0, b])$ by $Tf(x) = a + \int_0^x f(t) x e^{-xt} dt$. Prove that T is a contraction when $C([0, b])$ is equipped with the metric $d_\infty(f, g) = \max_{0 \le t \le b} |f(t) - g(t)|$. Use this fact to deduce that there is a unique solution $f \in C([0, \infty))$ to the integral equation $f(x) = a + \int_0^x f(t) x e^{-xt} dt$, $x \ge 0$.

Problem 4

An $n \times n$ real matrix A is said to be diagonally dominant if for each row the sum of the absolute value of off-diagonal terms in this row is strictly less than the value of the diagonal term in this row. Write $A = D - L - U$, where D is a diagonal matrix, L lower triangular and U upper triangular (both L and U with zero diagonal). Let the row-sum norm $\|X\|_\infty$ of a matrix $X = (X_{i,j})_{i,j=1}^n$ be defined by

$$\|X\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^n |X_{i,j}| ,$$

then show that if A is diagonally dominant, $\|L+U\|_\infty < \|D\|_\infty$. Consider $\mathbb{R}^n$ as a metric space with the (non-Euclidan) metric $d_\infty(x,y) = \max_{1\le i\le n} |x_i - y_i|$. If A is diagonally dominant, prove that for any $b \in \mathbb{R}^n$, a solution to the linear equation $Ax = b$ can be obtained as the limit of the sequence $(x^{(n)})_{n=1}^\infty$ defined by

$$x^{(n+1)} = D^{-1}(L+U)x^{(n)} + D^{-1}b$$

and this solution is unique. Thus, explain why any diagonally dominant $n \times n$ matrix A is invertible.