MATH 6360

Applicable Analysis
Fall 2021

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 2, due Thursday, September 9, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $p \in \mathbb{N}$, $b > 0$ and assume u is the solution of the integral equation

$$u(x) = \int_0^x \sin(u(t))(u(t))^p dt$$

on the interval $[-b, b]$.

a. Let $M = \sup_{-b \le x \le b} |u(x)|$. Prove that for each integer $n \ge 0$, $|u(x)| \le M^{np}|x|^n/n!$. Hint: $|\sin(y)| \le |y|$.

b. Use the preceding part to show that $u = 0$.

Problem 2

Consider the initial value problem with the differential equation $y'(x) = 1 + xy(x)$ and $y(0) = 0$.

a. Show that for any $0 < b < 1$, the integral operator T associated with this differential equation is a contraction mapping on $C([0, b])$, when we use the usual metric.

b. Show that there is a unique solution of this differential equation on $[0, b]$ for this initial value and *any* $b < \infty$. Hence deduce that there is a unique solution of the initial value problem on $[0, \infty)$.

Problem 3

Consider the initial value problem

$$y'(t) = t^2 + (y(t))^2, y(0) = 0$$

a. Show that for any $b > 0$, this differential equation satisfies a local Lipschitz condition (in the second variable) on the set $Q = [0, b] \times [-R, R]$, but not on the set $[0, b] \times \mathbb{R}$.

b. Integrate the inequality $y'(t) \ge 1 + (y(t))^2$ for $t \ge 1$ and use a monotonicity argument to prove that the solution to the initial value problem grows above any bound in finite time.

Problem 4

Let y be the solution to the initial value problem $y'(x) = e^{xy(x)}$ and $y(0) = 1$ for $x \in [0, 1/2]$. Suppose you wish to compare this with the solution y_n to the initial value problem $y'(x) = \sum_{k=0}^{n} \frac{(xy(x))^k}{k!}$, $y_n(0) = 1$, on $[0, 1/2]$.

1. Show that as $n \to \infty$, $y_n \to y$ uniformly on $[0, 1/2]$.
2. Find n so that $d_\infty(y, y_n) \equiv \max_{0 \le x \le 1/2} |y(x) - y_n(x)| < 0.0001$.