MATH 6360

Applicable Analysis
Fall 2021

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 3, due Thursday, September 16, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $a > 0$ and consider the integral equation for $f : [-a, a] \to \mathbb{R}$,

$$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} f(y) dy\,.$$

Use the contraction mapping theorem and a special starting point $f_0 \in C([-a, a])$ to show that the integral equation has a unique non-negative solution in $C([-a, a])$. Hint: Use $f_0(x) = 0$ and then find an inductive proof that the sequence $(f_n)_{n=0}^{\infty}$ associated with the integral operator only contains non-negative functions.

Problem 2

Let A be a $d \times d$ matrix such that there is $0 < r < 1$ and the linear map $T_A : \mathbb{R}^d \to \mathbb{R}^d$ given by matrix-vector multiplication $T_A : x \mapsto Ax$ satisfies $\|Ax - x\| \le r\|x\|$ for each $x \in \mathbb{R}^d$, where $\|a\|$ is the Euclidean length of the vector a. For fixed $y \in \mathbb{R}^d$, consider any x_0 and define a sequence $(x_n)_{n=0}^{\infty}$ by letting $x_{n+1} = x_n - Ax_n + y$. Explain why the sequence converges and if $x^* = \lim_{n\to\infty} x_n$, compute Ax^*.

Problem 3

Let y be a solution of the initial value problem $y'(x) = h(x, y(x))$ and $y(a) = y_0$, where h is continuous on $[a, b] \times \mathbb{R}$ and K-Lipschitz in the second variable. Assume η is a differentiable function satisfying $|\eta'(x) - h(x, \eta(x))| \le \epsilon$ for each $x \in [a, b]$ and $|\eta(a) - y_0| \le \delta$. Show that for $x \in [a, b]$,

$$|y(x) - \eta(x)| \le \delta e^{K(x-a)} + \frac{\epsilon}{K}(e^{K(x-a)} - 1)\,.$$

Hint: Find a variation of the proof for stability of solutions.

Problem 4

Let $h : [a, b] \times \mathbb{R}$ be a continuous function and for each fixed $x \in [a, b]$, $y \mapsto h(x, y)$ is non-increasing in y.

a. Let f and g be two solutions to the differential equation $y'(x) = h(x, y(x))$ with any (possibly different) initial values. Show that $\tau(x) = |f(x) - g(x)|$ is non-increasing in x. Hint: If $f(x) > g(x)$ on some interval I and $(x_1, x_2) \subset I$, express $f(x_2) - g(x_2) - (f(x_1) - g(x_1))$ as an integral.

b. Use the preceding part to show that if the initial value problem with $f(a) = y_0$, $y_0 \in \mathbb{R}$, has a solution on $[a, b]$, then it is unique. Hint: If there are two solutions f and g that are different, say $f(x_0) > g(x_0)$ for some $x_0 \in [a, b]$, then $\lim_{x\to a}(f(x) - g(x)) > 0$. This leads to a contradiction with $f(a) = g(a)$.