MATH 6360

Applicable Analysis

Fall 2021

First name: ____________ **Last name:** ______________ **Points:**

Assignment 4, due Thursday, September 23, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that the Hölder inequality for $f, g \in C([a,b])$ and $1 < p < \infty$ cannot be improved because for any $f \in C([a,b])$, there is $g \in C([a,b])$ such that $\int_a^b fg dx = \|f\|_p \|g\|_q$, $1/p + 1/q = 1$. Hint: For $x \in [a,b]$ with $f(x) \neq 0$, set $g(x) = |f(x)|^p / f(x)$ and if $f(x) = 0$ set $g(x) = 0$.

Problem 2

Show that if $1 \leq r < s$, then there is a constant $C_{r,s}$ such that if $f \in C([a,b])$, then $\|f\|_r \leq C_{r,s} \|f\|_s$. Hint: Use Hölder's inequality and note if $g(x) = |f(x)|^r$, $h(x) = 1$, then $\int_a^b |g(x)h(x)| dx = \|f\|_r^r$.

Problem 3

Recall that ℓ^p is the space containing each sequence $x = (x_j)_{j=1}^\infty$ for which $\sum_{j=1}^\infty |x_j|^p < \infty$. Show that if $1 \leq r < s$, then $\|x\|_s \leq \|x\|_r$. Hint: It is enough to show this for $\|x\|_r \leq 1$.

Problem 4

Recall that $[a,b] \subset \mathbb{R}$ is totally bounded, so it has a countable dense subset. Show that the space $C([a,b])$, equipped with d_∞, is separable, meaning it also has a countable dense subset. Hint: If function values are specified at a few points, then a function can be constructed by linearly interpolating.

Problem 5

By choosing balls centered in binary sequences, show the sequence space ℓ^∞ is not separable. You may quote that the set of binary sequences $S = \{a = (a_1, a_2, \dots) : a_j \in \{0,1\} \text{ for each } j\}$ is not countable, without proof.