MATH 6360

Applicable Analysis
Fall 2021

First name: ____________________ **Last name:** ______________________ | **Points:** |

Assignment 5, due Thursday, October 14, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let (X, d) and (Y, ρ) be metric spaces with completions (C, d) and (D, ρ), assuming $X \subset C$ and $Y \subset D$. Prove that the metric the space $(X \times Y, \sigma)$ with the metric $\sigma((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), \rho(y_1, y_2)\}$ has the completion $(C \times D, \sigma)$.

Problem 2

Let $c_{0,0}$ be the space of sequences that are eventually zero, so for each $x = (x_1, x_2, \dots) \in c_{0,0}$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = 0$. Equip this space with the metric d_∞. Show that the completion of $c_{0,0}$ is the space c_0, containing each sequence x with $\lim_{n\to\infty} x_n = 0$. Hint: You know $c_{0,0} \subset \ell^\infty$ and that ℓ^∞ is complete.

Problem 3

Let U be an open set in the interval $[a, b]$.

a. Show that the distance of any point x in U from the complement $U^c = [a, b] \setminus U$, given by $d(x, U^c) = \inf_{y \in U^c} d(x, y)$, is a continuous function on U. (Hint: U^c is closed.)

b. Show that the characteristic function χ_U of an open set $U \subset [a, b]$ is the (pointwise) limit of an increasing sequence of continuous functions. Here, $\chi_U(x) = 1$ if and only if $x \in U$ and otherwise $\chi_U(x) = 0$. Hint: Use the distance function to construct such a sequence, starting with $f_1(x) = \min\{1, d(x, U^c)\}$.

c. Use a result from class to deduce that the increasing sequence is Cauchy in $L^1([a, b])$ and hence that the characteristic function of any open set $U \subset [a, b]$ is in $L^1([a, b])$.

Problem 4

Define a map $T : C([0, 1]) \to C([0, 1])$ by

$$Tf(x) = \int_0^1 k(x, y) f(y) dy$$

where $k : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. Show that the operator norm $\|T\|$ equals

$$\|T\| \equiv \sup\{\|Tf\|_\infty : f \in C([0, 1]), \|f\|_\infty \leq 1\} = \max_{0 \leq x \leq 1} \int_0^1 |k(x, y)| dy \, .$$

Hint: If for $x \in [0, 1]$, $s(y) = 1$ if $k(x, y) > 0$, $s(y) = 0$ if $k(x, y) = 0$, and $s(y) = -1$ otherwise, then s is in $L^1([0, 1])$. Extending the integral to $L^1([0, 1])$ then gives $\int_{[0,1]} k(x, y) s(y) dy = \int_0^1 |k(x, y)| dy$.