MATH 6360

Applicable Analysis

Fall 2021

First name: ____________________ **Last name:** ________________________ **Points:**

Assignment 6, due Thursday, October 21, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $C([-1,1])$ be equipped with the norm $\|f\|_1 = \int_{-1}^{1} |f(x)|dx$. Show that the linear functional $F : f \mapsto f(0)$ is not bounded.

Problem 2

Show that if X and Y are normed spaces, and $B(X,Y)$ is a Banach space, then Y is a Banach space. Hint: If $F \in X^*$ is not the zero functional, consider $T : Y \to B(X,Y)$ given by $T(y)x = F(x)y$. Show T is linear and compare $\|y\|$ with $\|T(y)\|$.

Problem 3

Let c_0 be the normed vector space containing each sequence $x = (x_n)_{n\in\mathbb{N}}$ with $\lim_n x_n = 0$, equipped with the norm $\|x\|_\infty = \sup_n |x_n|$ for $x \in c_0$. Show that the dual space c_0^* is isometrically isomorphic to ℓ^1, the space of summable sequences, equipped with $\|y\|_1 = \sum_{n=1}^{\infty} |y_n|$ for $y \in \ell^1$.

Problem 4

Let $c_{0,0}$ be the space of real sequences with finitely many non-zero elements. Let for $x = (x_1, x_2, x_3, \dots)$ in $c_{0,0}$ its norm be $\|x\|_\infty = \max_n |x_n|$. Define for a fixed $x \in c_{0,0}$ a map $T_x : c_{0,0} \to \mathbb{R}$, $y \mapsto \sum_{n=1}^{\infty} x_n y_n$.

a. Show that for each x, T_x is continuous, that is, a bounded linear map.

b. Let $T : c_{0,0} \to c_{0,0}^*$ be defined by $T : x \mapsto T_x$. Show that if $A = \{x \in c_{0,0} : \|x\|_\infty \leq 1\}$, then for each $y \in c_{0,0}$, $\sup_{x\in A} |T_x y|$ is finite, but T is not uniformly bounded, so $\sup_{x\in A} \|T_x\| = \infty$.