MATH 6360

Applicable Analysis
Fall 2021

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 7, due Thursday, October 28, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that for $1 \leq p < s < \infty$, $\ell^p \subset \ell^s$ and the inclusion is strict, that is, there is $x \in \ell^s$ with $x \notin \ell^p$.

Problem 2

Let $(x^{(n)})_{n=1}^\infty$ be a sequence of elements in ℓ^1 such that for every $y \in \ell^\infty$, the sequence of numbers $(\langle x^{(n)}, y\rangle)_{n=1}^\infty$ is bounded, where $\langle a, b\rangle = \sum_{k=1}^\infty a_k b_k$. Show that there is $M > 0$ with $\|x^{(n)}\| \leq M$ for each $n \in \mathbb{N}$.

Problem 3

Show that if V is a proper subspace of a normed linear space X (meaning $V \neq X$), then V has empty interior. Hint: Having a non-empty interior means it contains a ball.

Problem 4

Let the "tent" functions be given by $\varphi_0^{(-1)}(x) = 1$, $\varphi_0^{(0)}(x) = x$, $\varphi_0^{(1)}(x) = \max\{1 - |1 - 2x|, 0\}$, and for $j \geq 2$, $\varphi_k^{(j)} = \varphi_0^{(1)}(2^{j-1}x - k)$, $k \in \{0, 1, \ldots, 2^{j-1} - 1\}$.

a. Draw the graphs of $\varphi_0^{(2)}$ and $\varphi_1^{(2)}$.

b. Show that if $f \in C([0,1])$ is given by a limit

$$f = a_{-1,0}\varphi_0^{(-1)} + a_{0,0}\varphi_0^{(0)} + \lim_{J\to\infty} \sum_{j=1}^{J} \sum_{k=0}^{2^{j-1}-1} a_{j,k}\varphi_k^{(j)}$$

where the convergence is with respect to d_∞, then each coefficient $a_{j,k}$ is uniquely determined. Hint: You may want to define $f_J = a_{-1,0}\varphi_0^{(-1)} + a_{0,0}\varphi_0^{(0)} + \sum_{j=1}^J \sum_{k=0}^{2^{j-1}-1} a_{j,k}\varphi_k^{(j)}$ and consider what properties the difference $f - f_J$ has, for example for which x we have $f(x) - f_J(x) = 0$.

c. Show that $\{\varphi_k^{(j)} : j \geq 0, 0 \leq k \leq 2^{j-1} - 1\}$ form a Schauder basis for $C([0,1])$, equipped with d_∞. The ordering for the elements in the Schauder basis is hereby assumed to be coming from the index $n = 2^{j-1} + k \in \mathbb{N} \cup \{1/4, 1/2\}$, so the elements in the Schauder basis are identified with the sequence $(\tilde{\varphi}_n)$ given by $\tilde{\varphi}_n \equiv \varphi_k^{(j)}$. Hint: You may quote results from an earlier homework without repeating the proof. Define a sequence of operators $(T_J)_{J=1}^\infty$ mapping $f \mapsto T_J f = a_{-1,0}\varphi_0^{(-1)} + a_{0,0}\varphi_0^{(0)} + \sum_{j=1}^J \sum_{k=0}^{2^{j-1}-1} a_{j,k}\varphi_k^{(j)}$ and appeal to the the uniform boundedness theorem.