MATH 6360

Applicable Analysis

Fall 2021

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Thursday, November 4, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let X be a Banach space, Y a normed vector space and $T : X \to Y$ bounded, linear. Assume there is $C > 0$ such that for each $x \in X$, $\|Tx\| \geq C\|x\|$. Show that the range $T(X)$ forms a complete subspace of Y and that the map $T' : X \to T(X), T'(x) = T(x)$ has a bounded inverse. (You may quote that a linear map T' has an inverse if and only if $T'x = 0$ implies $x = 0$.)

Problem 2

Consider $\mathbb{R}^d$ with the usual, Euclidean distance and norm. Show without using the open mapping theorem that each non-zero linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ which is onto is an open map. Hint: The unit sphere $\mathbb{S} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ is compact. Consider the function $f : \mathbb{S} \to \mathbb{R}, x \mapsto \|Tx\|$. Why is $\inf_{x\in\mathbb{S}} f(x) > 0$? Use this fact to deduce that T is open.

Problem 3

Let $X = c_{0,0}$, the space of sequences with finitely many non-zero elements, equipped with the norm from ℓ^∞. Let $T : X \to X$ be given by $(Tx)_k = x_k/k, k \in \mathbb{N}$. Show that T is a bijection, but that it does not have a bounded inverse.

Problem 4

Let $X = C([0, 1])$ be equipped with d_∞. We define $T : X \to X$ by $(Tf)(x) = \int_0^x f(t)dt$. Show that T is injective/one-to-one. Describe $T(X)$ in terms of properties of $g = Tf$. Does $T' : X \to T(X)$ have a bounded inverse?