Department of Mathematics University of Houston

MATH 6360
Applicable Analysis
Fall 2021

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 9, due Thursday, December 2, 11:30am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let $F : C([-1,1]) \to \mathbb{R}$ be given by $F(f) = \int_0^1 f(t)dt - \int_{-1}^0 f(t)dt$ where $C([-1,1])$ is equipped with d_∞. Let $Y = \ker F = \{f \in C([-1,1]), F(f) = 0\}$ and $h(x) = x$, then show that $\inf_{y\in Y} \|y - h\| = \frac{1}{2}$ but that there is no $z \in Y$ with $\|z - h\| = \frac{1}{2}$. Hint: How does $|F(f)|/\|F\|$ relate to the distance between f and Y?

Problem 2

Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L^q([a,b])$, let $T_g \in (L^p([a,b]))^*$ be given by $T_g f = \int_{[a,b]} fg dx$. Show that $T : g \mapsto T_g$ is an isometry from $L^q([a,b])$ to $(L^p([a,b]))^*$. Hint: It is enough to show this for a dense set, say for all $g \in C([a,b])$. You may quote Hölder's ineqality for L^p-spaces without proof.

Problem 3

Let for $j \in \mathbb{N}$, $a < b$, $x_0 = a, x_1 = a + (b-a)/2^j, \dots, x_m = a + (b-a)m/2^j, \dots, x_{2^j} = b$. Let for $f \in L^p([a,b])$, $T_j f(x) = 2^j \int_{[x_k, x_{k+1}]} f dx$ where $x_k \le x < x_{k+1}$, then as shown in class, $\|T_j f\|_p \le \|f\|_p$ and as $j \to \infty$, $T_j f \to f$. Show that for $F \in (L^p([a,b]))^*$, each $F_j : f \mapsto F(T_j f)$ is a bounded linear functional on $L^p([a,b])$ and as $j \to \infty$, $F_j \to F$. Hint: It is enough to show the convergence $F_j f \to Ff$ for f from a dense set.

For the next problem, you may quote the following fact (weak sequential compactness of the closed ball in $L^q([a,b])$: Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If $(f_j)_{j=1}^\infty$ is a bounded sequence in $L^q([a,b])$, so there is $M > 0$ such that for each $j \in \mathbb{N}$, $\|f_j\| \le M$, then there is a subsequence $(f_{j_n})_{n=1}^\infty$ and $f \in L^q([a,b])$ such that for each $g \in L^p([a,b])$, as $n \to \infty$, $\int_{[a,b]} g f_{j_n} dx \to \int_{[a,b]} g f dx$.

Problem 4

Show that the isometry $T : g \mapsto T_g$ from $L^q([a,b])$ to $(L^p([a,b]))^*$ is onto.