MATH 6361
Applied Analysis
Spring 2019

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 3, due Friday, February 8, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that that the polynomials $p_0(x) = 1/\sqrt{2}$ and for $n \in \mathbb{N}$

$$p_n(x) = \frac{1}{2^{n+1/2}} \frac{\sqrt{2n+1}}{n!} \frac{d^n}{dx^n}(x^2-1)^n$$

are what one would obtain from applying the Gram-Schmidt orthonormalization procedure to $1, x, x^2, \ldots$ in $L^2([-1,1])$. Explain why this is enough to conclude that $\{p_n\}_{n=0}^{\infty}$ is an orthonormal basis of $L^2([-1,1])$. You may use without proof that

$$\int_{-1}^{1} (x^2-1)^n dx = \frac{(-1)^n (n+1) 4^{n+1} (n!)^2}{(2(n+1))!} .$$

Problem 2

Prove that if $\frac{\alpha}{2\pi}$ is irrational, then for any 2π-periodic continuous function f on $\mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx .$$

Hint: First show this for the special case $f(x) = e^{imx}$, $m \in \mathbb{Z}$.

Problem 3

Prove that a bounded sequence $(x^{(n)})_{n=1}^{\infty}$ in ℓ^2 is weakly convergent to $x \in \ell^2$ if and only if for each $k \in \mathbb{N}$, the sequence of k-th entries $(x_k^{(n)})_{n=1}^{\infty}$ converges to x_k. Hint: Use that any bounded sequence has a subsequence that is weakly convergent.

Problem 4

Let A be a closed, bounded and convex set in a Hilbert space H and $f : H \to \mathbb{R}$ continuous, convex and bounded below on A, so $\inf_{x\in A} f(x) \in \mathbb{R}$. Show that f assumes its minimum on A.