MATH 6361

Applied Analysis

Spring 2019

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 4, due Friday, February 22, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that weak convergence of a sequence in the closed unit ball of a separable Hilbert space is equivalent to convergence with respect to a metric

$$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} |\langle x-y, f_j \rangle|$$

where $(f_j)_{j=1}^{\infty}$ is a suitable sequence of vectors.

Problem 2

Show that the linear map $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space is bounded if and only if for each weakly convergent sequence $(x_n)_{n=1}^{\infty}$ with weak limit x, $(Tx_n)_{n=1}^{\infty}$ is weakly convergent to Tx.

Problem 3

Show that for any bounded linear functional F on a real Hilbert space $\mathcal{H}$, the function

$$G(x) = \frac{1}{2}\|x\|^2 + F(x)$$

assumes its minimum at a unique vector x^* with $G(x^*) = \inf_{y \in \mathcal{H}} G(y)$.

Problem 4

Let $\epsilon > 0$ and

$$T_\epsilon = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}$$

define an operator on the Hilbert space $\mathbb{C}^2$, equipped with the standard inner product $\langle x, y \rangle = x_1\overline{y_1} + x_2\overline{y_2}$. Find the operator norm of T_ϵ and compare with $\max_{\|x\| \le 1} |\langle T_\epsilon x, x \rangle|$. Find T_ϵ^{-1}, compute its operator norm and compare with $\delta = \min_{\|x\|=1} |\langle T_\epsilon x, x \rangle|$.