Department of Mathematics University of Houston

MATH 6361

Applied Analysis

Spring 2019

First name: ____________________ **Last name:** ________________________ **Points:**

Assignment 5, due Friday, March 8, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Show that if A is a bounded, self-adjoint linear operator on a complex Hilbert space H and I the identity operator, then $T_\alpha = I + i\alpha A$ with $\alpha \in \mathbb{R}$ has a bounded inverse with norm $\|T_\alpha^{-1}\| \leq 1$.

Problem 2

Recall the real Sobolev space

$$H_0^2([-1,1]) = \left\{ f \in C([-1,1]) : f(x) = \int_{[-1,x]} g dt \text{ for } x \in [-1,1], g \in L^2([-1,1]), f(1) = 0 \right\}$$

equipped with the norm

$$\|f\|_{H_0^2} = \left(\int_{[-1,1]} |f|^2 dt + \int_{[-1,1]} |f'|^2 dt \right)^{1/2} .$$

Show that the linear functional $F : H_0^2([-1,1]) \to \mathbb{R}$ defined by

$$F(f) = f(0)$$

is bounded.

Problem 3

The numerical range of an operator T is the set $\rho(T) = \{\langle Tx, x\rangle, \|x\| = 1\}$. If T is bounded, linear and self-adjoint and $\rho_1 = \langle Tx, x\rangle < \rho_2 = \langle Ty, y\rangle$ with $\|x\| = \|y\| = 1$, then show $[\rho_1, \rho_2] \subset \rho(T)$. Hint: Consider the values of the quadratic form associated with T restricted to $(1-t)x + ty$, $t \in [0,1]$.

Problem 4

Let T be the operator on ℓ^2 defined by $(Tx)_k = x_k/k$. Show that if $(x^{(n)})_{n=1}^\infty$ is a sequence in ℓ^2 that converges weakly to 0, then $Tx^{(n)} \to 0$ in norm. Hint: Quote known results about such a sequence $(x^{(n)})_{n=1}^\infty$ in ℓ^2.