MATH 6361

Applied Analysis

Spring 2019

First name: ____________ **Last name:** ______________ **Points:**

Assignment 6, due Friday, March 29, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Let T be the operator on $L^2([-1,1])$ defined by

$$Tf(x) = \int_{[-1,1]} (x-y) f(y) dy \,.$$

Show that T is a Hilbert-Schmidt operator and compute its Hilbert-Schmidt norm $\|T\|_{HS}$. Hint: Consider an orthonormal basis of polynomials $\{1/\sqrt{2}, x/\sqrt{2/3}, \dots\}$.

Problem 2

Show that any Hilbert-Schmidt operator T on ℓ^2 can be approximated in operator norm by a finite-rank operator. Hint: Consider the canonical basis $(e_n)_{n=1}^\infty$ and define T_m for $m \in \mathbb{N}$ by

$$T_m x = \sum_{n=1}^m e_n \langle Tx, e_n \rangle \,.$$

Problem 3

If T is a bounded linear operator on a Hilbert space, prove that

$$\|TT^*\| = \|T^*T\| = \|T\|^2 \,.$$

Problem 4

Consider the operator T defined on $f \in C([0,1])$ by

$$Tf(x) = xf(x)$$

and continuously extended to all f in $L^2([0,1])$. Prove that T is self-adjoint, but not compact.

Problem 5

Consider the Hilbert space ℓ^2 with the canonical basis $(e_n)_{n=1}^\infty$ and the linear operator T satisfying

$$Te_k = \frac{1}{k} e_{k+1}$$

for each $k \in \mathbb{N}$. Show that T is compact, but that it has no eigenvectors.