MATH 6361

Applied Analysis
Spring 2019

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 8, due Friday, April 19, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Problem 1

Prove the following consequence of the Contraction Mapping Theorem: Let (X, d) be a complete metric space and $T : X \to X$ a continuous map such that the n-th iteration of the map is a contraction mapping, then T has a unique fixed point in X.

Problem 2

Consider the integral operator on $X = C([0, L])$ with $L > 0$ given by $Tf(x) = 1 + \int_0^x f(t)dt$ where the metric is obtained from the sup-norm.

a. Show that T is not a contraction mapping if $L > 1$.

b. Show that the n-th iteration of T satisfies

$$T^n f(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} + \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt$$

and use this to prove that for every $L > 0$, there is $n \in \mathbb{N}$ such that T^n is a contraction mapping on $C([0, L])$ and hence T has a unique fixed point.

Problem 3

Consider the Banach space $B(H)$ of bounded linear operators on a Hilbert space H, equipped with the operator norm. Let F be the map on $B(H)$ defined by $F(T) = T^2$. Show that F is differentiable and compute its derivative DF.

Problem 4

Let $X = L^p([0, 1])$ be the Banach space obtained from the completion of $C([0, 1])$ with respect to the p-norm, where $2 \le p < \infty$ and p is an integer, then $F(f) = \int_{[0,1]} (f(x))^p dx$ is differentiable on X, and compute its derivative DF. (Note: This includes showing that DF is a bounded linear functional.)

Problem 5

Let $X = Y = C([a, b])$, equipped with the sup-norm. Let K be a continuous function on $[a, b]^2$ and T be the non-linear integral operator

$$T(u)(x) = u(x) \int_a^b K(x, s)u(s)ds.$$

Show that T is differentiable and compute $DT(u)h$ for $u, h \in C([a, b])$.