Department of Mathematics University of Houston

MATH 6361

Applied Analysis

Spring 2019

First name: ____________________ **Last name:** ______________________ **Points:**

Assignment 9, Part I, due Friday, April 26, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Memorization

State the definition of differentiability at a point $x \in X$ for a map $F : X \to Y$ with Banach spaces X and Y.

Problem 1

For a bounded operator $A : H \to H$ on a Hilbert space H, explain why $e^A = \sum_{n=0}^{\infty} A^n/n!$ defines another bounded operator.

Problem 2

Let $B(H)$ be the space of bounded operators on H, equipped with the operator norm. Show that $F : A \mapsto e^A$ defines a map on $B(H)$ that is differentiable at $A = 0$.

Memorization

State the defining properties of a compact, normal operator T on a Hilbert space H.

Problem 3

Let $0 < r < 1$ be fixed. Define for $x, y \in [-\pi, \pi]$ the integral kernel $K(x,y) = \frac{1}{2\pi} \sum_{n=0}^{\infty} r^n e^{in(x-y)}$ and the associated integral operator

$$Tf(x) = \int_{[-\pi,\pi]} K(x,y) f(y) dy$$

for $f \in C([-\pi, \pi])$ that extends by (uniform) continuity to all of $L^2([-\pi, \pi])$. Show that T is a Hilbert-Schmidt operator on $L^2([-\pi, \pi])$ and compute its Hilbert-Schmidt norm $\|T\|_{HS}$.