MATH 6361

Applied Analysis

Spring 2019

First name: ____________________ **Last name:** ________________________ **Points:**

Assignment 9, Part II, due Friday, April 26, 10am

Please staple this problem sheet to your homework. When asked to prove something, make a careful step-by-step argument. You can quote anything we covered in class in support of your reasoning.

Memorization

State the inverse function theorem between Banach spaces.

Problem 1

Let $X = C([a,b])$, equipped with the sup-norm, $U = \{f \in X : f(x) \neq 0 \text{ for } x \in [a,b]\}$ and $F : U \to X$ be given by $F(f)(x) = 1/(f(x))$.

a. Let $g \in C([a,b])$ and consider the multiplication operator M_g on X, $M_g : f \mapsto gf, (gf)(x) = g(x)f(x)$. Explain briefly why $\|M_g\| = \|g\|_\infty$ and show that if $g(x) \neq 0$ for each $x \in [a,b]$, then M_g has a bounded inverse.

b. Show that the map F is a C^1-function. Compute $DF(f)$ at $f \in U$ given by $f(x) = 1, x \in [a, b]$. What can you conclude about the inverse of F?

Memorization

Let $S(q) = \int_a^b L(t, q(t), q'(t))dt$ be a real-valued functional, with a C^2-function L on $\mathbb{R}^3$. Let the domain of S be the affine subspace $\mathcal{A} = \{q : q \in C^2([a, b]) : q(a) = q_1, q(b) = q_2\}$. State an ordinary differential equation satisfied by $q \in \mathcal{A}$ if q is a critical point of $S : \mathcal{A} \to \mathbb{R}$.

Problem 3

Let $L(t, q, v) = \frac{1}{2}v^2 - V(q)$ with a C^1-function V. What is the ordinary differential equation satisfied by q if it is a critical point for S on the domain $\mathcal{A}$?

Problem 4

Let $\mathcal{A}$, S, L and T be as above and $V(q) = gq$, with a constant $g > 0$. Derive a differential equation for q which is a critical point of S and find the solution with boundary conditions $q(0) = 0$ and $q(1) = 0$. (You do not need to show uniqueness.)