REVIEW: THE CONTRACTION MAPPING THEOREM

We will study how contraction mappings provide a method to prove existence and uniqueness of fixed points and also a method to find the fixed point, at least approximately.

Definition 1. *Let (X, d) be a metric space. A function $f : X \to X$ is called a* **contraction mapping** *provided that there is a number $r, 0 < r < 1$, so that $d(f(x), f(y)) \leq rd(x, y)$ for every $x, y \in X$.*

Recall that saying that f is a contraction mapping is the same as saying that it is Lipschitz continuous with constant $r < 1$.

Definition 2. *Given a map $f : X \to X$, any point $x^* \in X$ satisfying $f(x^*) = x^*$ is called a* **fixed point** *of f.*

Theorem 3 (Contraction Mapping Principle)**.** *Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction mapping with Lipschitz constant $r < 1$, then*

1. *there exists a unique point $x^* \in X$, such that $f(x^*) = x^*$,*
2. *if $x_0 \in X$ is any point and we define a sequence inductively by setting $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = x^*$,*
3. *for this sequence, we have that $d(x_n, x^*) \leq \frac{d(x_0,x_1)r^n}{1-r}$.*

Proof. First, we show that the inductively defined sequence converges. To see this, since X is complete, it is enough to show that the sequence is Cauchy.

Let $A = d(x_0, x_1)$. Then we have that $d(x_1, x_2) = d(f(x_0), f(x_1)) \leq rd(x_0, x_1) = rA$. Similarly, $d(x_2, x_3) = d(f(x_1), f(x_2)) \leq rd(x_1, x_2) \leq r^2A$. By induction, we get that $d(x_n, x_{n+1}) \leq r^nA$.

Next, given $m > n$, then by the triangle inequality $d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m)) \leq \sum_{k=n}^{m-1} r^kA \leq \frac{Ar^n}{1-r}$. Given any $\epsilon > 0$, we may choose an integer N such that $\frac{Ar^N}{1-r} < \epsilon$. Then if $m, n \geq N$, we have that $d(x_n, x_m) < \epsilon$. Thus, $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Let $x^* = \lim_n x_n$. Then since f is continuous, $f(x^*) = \lim_n f(x_n) = \lim_n x_{n+1} = x^*$, so x^* is a fixed point for f.

Now if we fix any n, then $d(x^*, x_n) = \lim_m d(x_m, x_n) \leq \frac{Ar^n}{1-r}$, by the above estimate, which proves (3).

We now know that there is a point x^*, with $f(x^*) = x^*$ and that the inductively defined sequence converges to it. What remains is to show that x^* is a unique fixed point. To complete the proof of the theorem, we show that if $f(x') = x'$, then $x' = x^*$ To see this last fact, note that $d(x', x^*) = d(f(x'), f(x^*)) \leq rd(x', x^*)$. Since $r < 1$, this implies that $d(x', x^*) = 0$. □