Practice Final Exam – Math 6360
May, 2019

First name: ________________ Last name: ____________________ Last 4 digits ID: _____________

1 Memorization

a. State the polarization identity for the inner product of two vectors x and y in a complex Hilbert space.

b. State the Riesz representation theorem for linear functionals on Hilbert spaces.

c. Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ in a Hilbert space to have a weak limit and explain why this weak limit, if it exists, is unique.

d. State a condition for uniform convergence of Fourier series in terms of the function for which the Fourier series is computed.

Turn in this first part to obtain the next portion of the exam.

First name: ________________ Last name: ____________________ Last 4 digits ID: _____________

In the following problems, you may quote statements from class to simplify your answers. You do not need to give a proof of a statement if it was discussed in class.

2 Problem

Prove that a vector x in a Hilbert space H is orthogonal to all the vectors in a closed subspace Y if and only if $\|x\| \leq \|x - y\|$ for each $y \in Y$.

3 Problem

In this problem, all function spaces are over the complex numbers. For $u \in C([0,1])$, define

$$Tu(x) = \int_0^{1-x} u(y)dy$$

a. Explain why the linear map T on $C([0,1])$ extends uniquely to a map $T : L^2([0,1]) \to L^2([0,1])$ which is bounded.

b. Show that this bounded linear map T on $L^2([0,1])$ is self-adjoint. Hint: It is enough to show that for each $u, w \in C([0,1])$, $\langle Tu, w\rangle = \langle u, Tw\rangle$.

c. Show that T is a compact operator.

d. Let $f \in C([0,1])$. Show that there is a unique solution $u \in L^2([0,1])$ to the equation

$$u + iTu = f\,.$$

e. Explain why the solution to $u + iTu = f$ is actually a function $u \in C([0, 1])$.

4 Problem

Prove that T is a compact linear operator on an infinite-dimensional, separable Hilbert space and $(e_n)_{n=1}^{\infty}$ an orthonormal basis, then $Te_n \to 0$.

5 Problem

Let A be a self-adjoint compact operator on a separable, real Hilbert space H and $Q : x \mapsto \langle Ax, x \rangle \geq 0$ the non-negative quadratic form associated with A.

a. Find the derivative $DQ(x) : H \to \mathbb{R}$ at a fixed $x \in H$. You do not need to prove differentiability.

b. Show that for any $x, h \in H$, Q satisfies $Q(x + h) \geq Q(x) + DQ(x)h$.

6 Problem

Let f_1, f_2 and f_3 be 3 real-valued functions in $C([a, b])$ and for any real-valued $g \in C([a, b])$, $F(g) = \sum_{k=1}^{3} \int_{[a,b]} (g(x) - f_k(x))^2 dx$, where the norm is the usual Euclidean norm. Show that F is differentiable and compute the derivative $DF(g)$. Find all critical points for the function F.

7 Problem

Let $\mathcal{T}([-\pi,\pi])$ be space of complex trigonometric polynomials, so $\mathcal{T}([-\pi,\pi]) = \mathrm{span}\{u_k : k \in \mathbb{Z}\}$ with $u_k(x) = e^{ikx}$. Let $\mathcal{T}([-\pi,\pi])$ be equipped with the inner product

$$\langle f, g\rangle_{\mathcal{T}} = \int_{[-\pi,\pi]} f\overline{g}dx + \int_{[-\pi,\pi]} f'\overline{g'}dx\,.$$

a. Show that the functions $(u_n)_{n\in\mathbb{Z}}$ given by $u_n(x) = e^{inx}$ form an orthogonal system and compute the norm $\|u_n\|_{\mathcal{T}}$. Deduce factors $c_n \in \mathbb{R}$ such that $e_n = c_n u_n$ defines an orthonormal system $(e_n)_{n\in\mathbb{Z}}$.

b. Explain why if $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{T}([-\pi,\pi])$, then the derivatives $(f_n')_{n=1}^{\infty}$ form a Cauchy sequence in $L^2([-\pi,\pi])$.

c. Use the result from the last part to conclude that for $\epsilon > 0$ and $f \in C([-\pi,\pi])$ with $f(\pi) = f(-\pi)$ and

$$f(x) = f(-\pi) + \int_{[-\pi,x]} g\,dy,\; g \in L^2([-\pi,\pi]),$$

there exists a trigonometric polynomial q such that

$$\sup_{x\in[-\pi,\pi]} |f(x) - q(x)| < \epsilon\,.$$

Hint: First prove $\|f_0 - q_0\|_\infty < \epsilon$ where f_0 and q_0 have the additional property $\int_{[-\pi,\pi]} f_0\,dx = \int_{[-\pi,\pi]} q_0\,dx = 0$.

[empty page]